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Froude Number Dependence of Gravity Wave Radiation from Unsteady Rotational Flow in f -plane Shallow Water System

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Continuous gravity wave radiation from unsteady rotational flow is investigated numerically in f -plane shallow water system for a wide range of Rossby number, Ro , and Froude number, Fr , using forced dissipative system. For large Ro , where the effect of the earth rotation is negligible, gravity wave flux is proportional to Fr , which is consistent with the aero-acoustic sound radiation theory (the Lighthill theory). However, this power law is not valid for relatively small Ro , where the effect of the earth rotation is important. The mechanism for the breakdown of the power law is also examined using temporal spectral analyses.

1. INTRODUCTION

Gravity waves play a fundamental role in driving the general circulation of the middle atmosphere to transport significant amount of energy and momentum. However, the process of gravity wave radiation remains an open problem, especially concerning non-orographic waves. Several observational studies suggest gravity wave radiation from strong rotational jet flows¹⁾.

As a first step to investigate the radiation process of gravity waves, Ford²⁾ used f -plane shallow water system which is the most simplified system in which both balanced rotational modes and unbalanced gravity wave modes can exist. He investigated initial gravity wave radiation from unstable vortex train, and pointed out the dependence on Froude number, Fr , based on an analogy with the aero-acoustic sound radiation theory³⁾. However, his study covered limited Ro parameter values only and using non-forcing system. Therefore, in this study, we adopt forced dissipative system and investigate continuous gravity wave radiation from unsteady rotational flow. Here, we study the dependence of gravity wave flux on Fr . The effect of the earth rotation on the gravity wave radiation is investigated as well.

2. MODEL DESCRIPTION

The basic equations used in this study are shallow water equations on an f -plane with forcing and dissipation. For a plane rotating with angular velocity $f/2$, where the external parameter f is the Coriolis parameter, which is generally called f -plane, the equations are as follows

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -g \frac{\partial h}{\partial x} - \underbrace{\alpha(u - u_0)}_{\text{forcing}}, \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -g \frac{\partial h}{\partial y} - \underbrace{\alpha(v-0)}_{\text{forcing}}, \tag{2}$$

$$\frac{\partial h}{\partial t} + u \frac{\partial(h-h_b)}{\partial x} + v \frac{\partial(h-h_b)}{\partial y} + (h-h_b) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \tag{3}$$

where, dependent variables u and v are the longitudinal (x) and the latitudinal (y) velocities, respectively, and h is the surface level. $u_0(y)$ is the basic state of u for the zonal jet defined below and $h_b(y)$ is the bottom topography. External parameter g is the gravitational acceleration. The terms with relaxation parameter α mean the effect of forcing to maintain an unstable jet. This forcing is added near the jet region only.

The basic state of the zonal jet, which is barotropically unstable is prescribed as⁴⁾

$$u_0(y) = U_0 \operatorname{sech} \left\{ \frac{2(y-y_0)}{B} \right\}, \tag{4}$$

where U_0 , y_0 , and B are parameters to determine the intensity, the position, and the width of the jet, respectively. In this study, we introduce the bottom topography, h_b , so that the depth of the fluid for the basic state is constant, that is, $h_0 - h_b = H_0$ (const.). Here, h_0 is the surface level for the basic state which satisfy the following equation $fu_0 = -gdh_0/dy$. By introducing this bottom topography, radiated gravity wave is symmetric with respect to the center of the jet. Fig. 1 shows the latitudinal profiles of the zonally symmetric basic state of the zonal flow $u_0(y)$ and the latitudinal gradient of the potential vorticity $dq_0(y)/dy$ ($q_0 = (\zeta_0 + f)/H_0$; $\zeta_0 = -du_0/dy$ is the vorticity for the basic state) for $Ro = 100$ and $Fr = 0.3$.

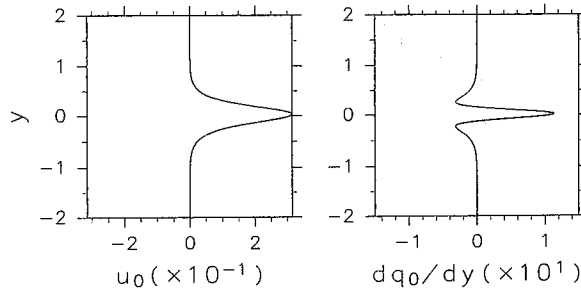


Fig. 1 Latitudinal profiles of the zonally symmetric basic state ($Ro = 100$ and $Fr = 0.3$). (left) the zonal flow $\bar{u}(y)$, and (right) the latitudinal gradient of the potential vorticity $dq_0(y)/dy$.

In this study, the width B , the intensity U_0 , and the depth of the basic state H_0 are used as the length, the speed, and the depth scales, respectively. Then the non-dimensional parameters Ro (Rossby number) and Fr (Froude number) are determined as

$$Ro \equiv \frac{U_0}{fB}, \quad Fr \equiv \frac{U_0}{\sqrt{gH_0}}. \tag{5}$$

In this study, we fix U_0 and B for the numerical experiments. Then Ro determines f , and Fr determines $1/\sqrt{gH_0}$, respectively. We investigate gravity wave radiation from unsteady rotational flow for a wide range of Ro and Fr .

The domain for the numerical experiments is set to be periodic in the longitudinal (x) direction, and infinite in the latitudinal (y) direction. Calculation for the y direction is performed by

using the mapping $y = 2r \tan(\theta/2)$ so that $\theta \in (-\pi, \pi)$ is projected to $y \in (-\infty, \infty)$ ⁵⁾. Here, $r (= 4)$ means the scaling parameter to determine the ratio of grid intervals for the x direction and the y direction. Using this projection, we can save the number of grids of the propagation region of gravity waves, while taking the grid intervals of the jet region densely. The mapped shallow water equations are computed using a spectral transform method⁶⁾. Resolution is set to be $K = 21, L = 336$ (64×1024 grids), where K and L are the truncation wave numbers for the x direction and the θ direction, respectively. The 4th order Runge-Kutta method is used for the time-integrations. We also introduce a pseudo artificial viscosity term to smooth numerical behavior, which also acts as an absorber of gravity waves far from the jet region.

3. THE Lighthill EQUATION AND THE F_r DEPENDENCE OF GRAVITY WAVE FLUX

Source of gravity wave is introduced on the analogy with the aero-acoustic sound radiation theory (the Lighthill theory). Rewriting the shallow water equations (1)–(3) in the flux form, we obtain the Lighthill equation in the forced-dissipative system with topography as follows

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - c_0^2 \nabla^2 \right) \frac{\partial h}{\partial t} = \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} - F_1 - F_2, \quad (6)$$

where $x_1 = x, x_2 = y$ and $c_0 = \sqrt{gH_0}$ is a phase velocity of the fastest gravity wave. The left hand side means gravity wave propagation, and the right hand side means the source of gravity waves. T_{ij} is written as

$$T_{ij} = \frac{\partial(hu_i u_j)}{\partial t} + \frac{f}{2} (\epsilon_{ik} h u_j u_k + \epsilon_{jk} h u_i u_k) + \frac{g}{2} \frac{\partial}{\partial t} (h - H_0)^2 \delta_{ij}. \quad (7)$$

Here, $\epsilon_{12} = \epsilon_{21} = -1, \epsilon_{11} = \epsilon_{22} = 0$ and $u_1 = u, u_2 = v$. F_1 and F_2 are the source terms related to the zonal forcing and the bottom topography, respectively. The effects of these source terms are negligible, since the gravity waves are mainly radiated from vortex motion. Therefore, we will omit the contribution from these terms in the following part of this paper.

Averaging (6) in the x direction, we obtain the following equation,

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - c_0^2 \frac{\partial^2}{\partial y^2} \right) \frac{\partial \bar{h}(y, t)}{\partial t} = \frac{\partial^2}{\partial y^2} \bar{T}_{22}(y, t), \quad (8)$$

where $\bar{(\cdot)}$ means x -averaged value. From (8), we can estimate the gravity wave amplitude far from the jet region as follows²⁾. Assuming that the first term is dominant in (7) and the source term depends on t sinusoidally, \bar{T}_{22} is scaled as,

$$\bar{T}_{22}(y) \sim \omega \bar{u}^2(y) H_0, \quad (9)$$

where ω is the frequency of the source and \bar{u}^2 is the Fourier transform of u^2 with respect to t . Here ω is mainly determined by the frequency of the unstable vortex motion. We also assume that the source is localized near the jet region, (8) is scaled as,

$$i\omega \bar{h}(y) \sim \frac{(\omega^2 - f^2)^{\frac{1}{2}}}{2c_0} \left(e^{i(\omega^2 - f^2)^{\frac{1}{2}} y / c_0} \right) \int_{-B_j}^{B_j} \frac{1}{c_0^2} \bar{T}_{22}(y) dy, \quad (10)$$

where, B_j is the width of the jet region. Then, using the scale for \bar{T}_{22} from (9), we obtain

$$\bar{h}(y, t) \sim H_0 \left(\frac{U_0}{c_0} \right)^2 \left(\frac{(\omega^2 - f^2)^{\frac{1}{2}}}{c_0} B \right). \quad (11)$$

In the shallow water equations, pseudo-energy A_e and its flux \mathbf{F}_e where the value of the potential vorticity is zero are as follows²⁾,

$$A_e = \frac{1}{2}h(u'^2 + v'^2) + u_0u'h + \frac{1}{2}gh'^2, \quad (12)$$

$$\mathbf{F}_e = \mathbf{u}A_e + H_0\mathbf{u}_0 \cdot \mathbf{u}'\mathbf{u}' - \frac{1}{2}H_0\mathbf{u}'^2\mathbf{u}_0 + \frac{1}{2}gh'^2\mathbf{u} + gH_0h'\mathbf{u}'. \quad (13)$$

Here, $(u, v, h) = (u_0, 0, H_0) + (u', v', h')$ and $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, $\mathbf{u}_0 = (u_0, 0)$, $\mathbf{u}' = (u', v')$. Therefore, using the scaling for h' from (11), letting $h' = \bar{h}$ and assuming that the second term in (12) and the first term in (13) are dominant, the pseudo-energy flux far from the jet region is scaled as

$$\|\mathbf{F}_e\| \sim \frac{\omega B^2 U_0^3 Fr (\omega^2 - f^2)^{\frac{1}{2}}}{g}. \quad (14)$$

Since we fix B, U_0 in this experiment, if ω is constant and f is negligible, which corresponds to no earth rotation, (14) shows that gravity wave flux F_e is proportional to Fr . If $\omega B \sim U_0$ and H_0 is fixed as in the case of the aero-acoustic sound radiation theory, F_e is proportional to the sixth power of U_0 .

4. RESULTS

Fig. 2 shows the t - y cross section of $\partial\bar{h}/\partial t$ ($5.0 \leq t \leq 15.0$, $-50 \leq y \leq 50$) for $Ro = 100$, $Fr = 0.3$. $\partial\bar{h}/\partial t$ is a good index of gravity wave, since $\partial\bar{h}/\partial t$ indicates linear gravity wave propagation if there is no source term as shown in (6). As the jet fluctuates quasi-periodically in time, gravity waves are radiated from the jet region (around $y = 0$) continuously with the phase speed of the fastest gravity wave, $\sqrt{gH_0} = 100\pi/3$. Unlike classical Rossby adjustment problem which supposes the initial unbalanced state, gravity waves are continuously radiated from nearly balanced rotational flow region, where Fr is so small that balance dynamics is thought to be good approximation for the full system.

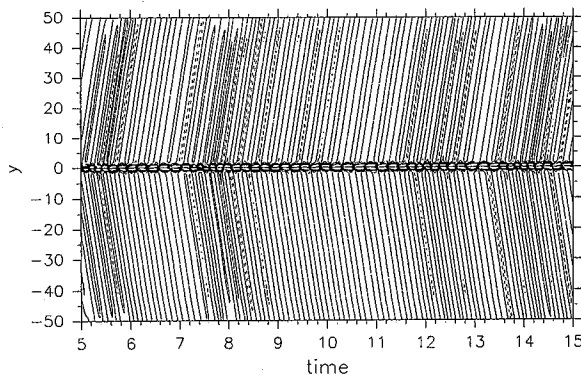


Fig. 2 The t - y cross section of $\partial\bar{h}/\partial t$ for $5.0 \leq t \leq 15.0$ and $-50 \leq y \leq 50$. Contour interval is 1. Solid contour lines correspond to positive values and 0, while broken contour lines correspond to negative values.

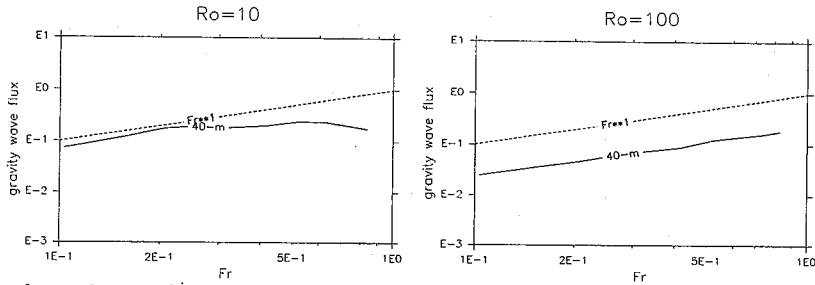


Fig.3 Fr dependence of zonally averaged gravity wave flux at $y = 40$ for $Ro = 10$ (left) and $Ro = 100$ (right). The value of the flux is time-averaged for $5.0 \leq t \leq 15.0$. The dashed lines indicate Fr^1 .

We investigated this gravity wave radiation for several different parameter values ($Ro = 1, 3, 5, 7, 10, 30, 50, 70, 100, 500, 1000$ and $Fr = 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$). Though time evolution of flow field is qualitatively similar for different parameter values, gravity wave radiation for each experiment has several different features. Fig. 3 shows the Fr dependence of gravity wave flux for $Ro = 10$ and $Ro = 100$. For large $Ro (= 100)$, where the effect of the earth rotation is negligible, gravity wave flux is proportional to Fr , which is consistent with the scale analysis based on the aero-acoustic sound radiation theory (14). However, the dependence on Fr is different for relatively small $Ro (= 10)$, where the effect of the earth rotation is important. For large $Fr (\geq 0.5)$, gravity wave flux is not proportional to Fr .

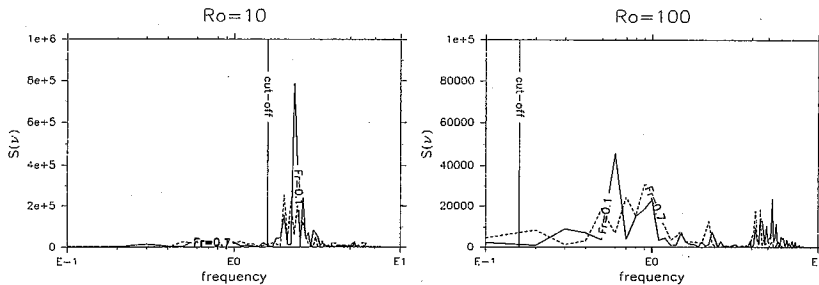


Fig.4 The frequency spectrum of zonally averaged source integrated in the jet region ($\int_{-8}^8 \overline{T_{22}} dy$) for $Ro = 10$ (left) and $Ro = 100$ (right). Solid line is for $Fr = 0.1$, while broken line is for $Fr = 0.7$. The spectra are rescaled for each Fr , considering the dependence on Fr of gravity wave source ($\overline{T_{22}} \propto Fr^{-2}$).

To investigate why the Fr power law is not valid for $Ro = 10$, the frequency spectrum of gravity wave source integrated in the jet region is shown in Fig. 4. The inertial frequency $f/2\pi$ denoted as "cut-off" means critical frequency for gravity waves. Gravity waves can propagate only if the frequency of gravity wave exceeds this inertial cut-off frequency. For both Ro cases, the amplitude for $Fr = 0.7$ is smaller than that for $Fr = 0.1$ at high frequency ($2 \leq \nu \leq 10$; ν is the frequency) on average. The reason is as follows. Since the deformation radius is smaller for large Fr , the interaction between vortices is inhibited and the vortices become more steady. This decrease of the amplitude at higher frequency does not affect gravity wave flux for large $Ro (= 100)$. This is because the inertial cut-off frequency is so low that gravity waves can be radiated from unsteady

rotational flow at low frequency ($\nu \leq 2$). On the other hand, for small $Ro (= 10)$, since the inertial cut-off frequency is relatively high, gravity waves are not radiated from unsteady rotational flow at lower frequency. Therefore, the decrease of the amplitude at higher frequency directly affects gravity wave radiation and causes the breakdown of Fr power law for small $Ro (= 10)$.

5. SUMMARY

Using forced dissipative system, gravity wave radiation from unsteady rotational flow is investigated numerically in f -plane shallow water system for a wide range of Ro and Fr for the first time. Gravity wave flux is proportional to Fr for large $Ro (= 100)$, which is in agreement with the aero-acoustic sound radiation theory. On the other hand, this Fr dependence does not hold for relatively small $Ro (= 10)$. We investigate the frequency spectral of gravity wave source and show that vortices become more steady for large Fr . This is because deformation radius is smaller for large Fr , thus interaction between vortices become more difficult. Since the inertial cut-off frequency is relatively high for small Ro , gravity waves of lower frequency are not radiated from this rotational flow. This is why Fr power law of gravity waves flux is not valid for small Ro .

We have shown that gravity waves are continuously radiated from unsteady rotational flow. Though our model is too simple to apply to the real atmosphere, the results suggest that the gravity waves can be radiated from jet regions in the real atmosphere, even in the parameter range in which the geostrophic balance is thought to be valid. Furthermore, since the shallow water system is introduced on the assumption of strong stratification, this system has external gravity waves only which propagate in the horizontal direction. On the other hand, in the real atmosphere, there are internal gravity waves which propagate both in the horizontal and in the vertical direction. These internal gravity waves can be radiated more easily from unsteady jet flows. It will be also interesting to explore the possibility of similar Fr dependence of gravity wave radiation in the real atmosphere.

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