

A Relationship between Local Error Growth and Quasi-stationary States: Case Study in the Lorenz System

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ABSTRACT

Properties of the local predictability in the Lorenz system of three variables are investigated as a first step to develop a dynamical method for the skill prediction in the numerical weather forecasts instead of conventional statistical and empirical methods. As a measure of the local predictability, we adopt Lorenz's index which gives the amplification rate of the root-mean-square error during a prescribed time interval. In particular, we exert ourselves to understand a role of the quasi-stationary state in determining the variation of the Lorenz index.

In an intermittent chaos regime, the Lorenz index determined for a time interval of the one return in the Poincaré section has a minimum value at the onset of the laminar phase, gradually increases during the laminar phase, and abruptly attains a large value at the break of the laminar phase. If we consider the laminar phase as a quasi-stationary state generated by a local minimum point in the one-dimensional Poincaré map, this characteristic evolution of the Lorenz index is directly connected with the local dynamics of the local minimum point.

The fine phase-spatial distribution of the Lorenz index for a short time interval on the Lorenz attractor is also discussed in connection with the role of the unstable stationary point in organizing the local predictability.

1. Introduction

Recent progress of the theory of nonlinear dynamical systems has been providing several new methods to facilitate the description and understanding of aperiodic low-frequency variations in large-scale atmospheric motions, to which the conventional linear theories cannot be applied. Among others, the concept of *deterministic chaos* introduced by Lorenz (1963) is one of the most attractive paradigms for the large-scale atmospheric dynamics. In this paradigm, the temporal variation of atmospheric motions is considered to be a trajectory embedded in a *strange attractor* in phase space.

There are two complementary approaches to pursue this new paradigm. One is to grasp the global, or gross feature of the strange attractor. Fraedrich (1986, 1987) and Keppenne and Nicolis (1989) pointed out a possibility of the existence of a low-dimensional strange

attractor in atmospheric motions by obtaining the correlation dimension (Grassberger and Procaccia 1983a,b) with a time series of the atmospheric dataset. However, the obtained low-dimensionality has not been commonly convinced as yet because of the following problems: 1) the time series is much too short for a reliable estimate of the dimension; 2) the convergence of the dimension is not clear. On the other hand, the Kaplan-Yorke estimate is much more robust when the Lyapunov exponents can be computed. Yano and Mukougawa (1991) directly determined the Kaplan-Yorke dimension of an attractor in a two-layer, quasi-geostrophic model. The obtained dimension is not a small number even in this idealized simple model.

The other approach is to extract local, or fine characteristics of the strange attractor. The atmospheric motions occasionally show recurrent quasi-stationary states, such as the blocking phenomena in the mid-latitude troposphere, in the course of the evolution. This could be caused by some inhomogeneous local structures of the attractor. By using a truncated barotropic model on a sphere, Legras and Ghil (1985) classified the quasi-stationary states into several *weather*

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regimes and demonstrated that these recurrent quasi-stationary states occur in the vicinity of unstable stationary points in phase space. An unstable stationary point attracts trajectories repeatedly along its stable manifold and repels them along its unstable manifold. In this sequence, a quasi-stationary state appears repeatedly in chaotic motions. Moreover, Mukougawa (1988) found that the quasi-stationary states are generated by a local minimum point; the evolution of trajectories slows down in the vicinity of the local minimum point in phase space. The local minimum point is defined as a point at which the speed of the trajectory in phase space has a local minimum value. In a bifurcation diagram, the local minimum point branches off from the limit point of a stationary point.

This recognition of the atmospheric motion as a deterministic chaos has been also closely connected with the theoretical studies of the atmospheric predictability. Since the pioneering work by Lorenz (1963), several efforts have been devoted to estimate the predictable time scale of the atmospheric motion by using several numerical models (e.g., Smagorinsky 1969; Lorenz 1982). The time scale corresponds to the long-period average of the divergence rate between initially adjacent trajectories on the attractor. This measure reflects the global structure of the attractor.

The temporal variation of the predictability has been a central issue in the medium and extended range numerical weather prediction. Because forecast models show considerable variability of the predictability (Palmer and Tibaldi 1988), the prediction of the forecast skill gives important information to the numerical weather forecasts. Although there are several works on the operational skill forecast, by utilizing empirical or statistical indices such as the persistency of the forecast and the spread between adjacent forecasts (e.g., Kalnay and Dalcher 1987; Palmer and Tibaldi 1988), the dynamical aspects of this temporal variation of the predictability have not been addressed as yet.

Variation of the predictability could result from nonuniform distribution of the local divergence rate of adjacent trajectories on the attractor. The variation might be connected with the local properties of the attractor, such as the quasi-stationary state. In this paper, as a first step to understand the dynamics of the local predictability, we examine the property of the local error growth in the Lorenz system (Lorenz 1963) of three variables. In section 2, we define the local minimum point which determines local characteristics of the attractor and the Lorenz index as a measure of the local error growth. In section 3, we investigate the local dynamics of the error growth in the intermittent chaos regime. In section 4, we discuss the role of the unstable stationary point in determining the phase-spatial distribution of the Lorenz index, and also imply the possibility of the application of our results to the operational numerical weather prediction. Conclusion is in section 5.

2. Definition

a. Local minimum points

Let us consider a dynamical system for which time evolution is described by a set of nonlinear ordinary differential equations in n -dimensional phase space:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mu), \quad (1)$$

where $\mathbf{x}(t)$ is an n -dimensional variable vector and μ the bifurcation parameter. First, let the transiency index $T(t) = T(\mathbf{x}(t), \mu)$ be defined as

$$T(\mathbf{x}, \mu) = \|\mathbf{f}(\mathbf{x}, \mu)\|, \quad (2)$$

which measures the speed of the evolution in phase space. Mukougawa (1988) defined the local minimum point as a point at which $T(\mathbf{x}, \mu)$ has a local minimum value in phase space. The stationary point is a subset of the local minimum points.

b. The Lorenz index: a measure of local error growth rate

We introduce Lorenz's index (Lorenz 1965), which is a measure of the local error growth rate. The time evolution of an infinitesimally small error $\mathbf{y}(t)$ superposed on the trajectory $\mathbf{x}(t)$ of Eq. (1) obeys the following set of tangential linear differential equations:

$$\dot{\mathbf{y}}(t) = \mathbf{J}(t)\mathbf{y}(t), \quad (3)$$

where

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (4)$$

is the Jacobian matrix. The solution of Eq. (3) can be written as

$$\mathbf{y}(t + \tau) = \mathbf{A}(t + \tau, t)\mathbf{y}(t), \quad (5)$$

where $\mathbf{A}(t + \tau, t)$ is an $n \times n$ matrix and called the error matrix by Lorenz (1965); the evolution of an arbitrary error between the time interval from t to $t + \tau$ is described by the error matrix \mathbf{A} .

Lorenz (1965) examined the amplification rate of the root-mean-square (rms) error between time t and $t + \tau$. Let us consider the evolution of initial errors on a surface of the n -dimensional sphere with a radius of ϵ :

$$\mathbf{y}^T(t)\mathbf{y}(t) = \epsilon^2. \quad (6)$$

According to Eq. (5), this sphere is evolved into an ellipsoid at $t = t + \tau$:

$$\mathbf{y}^T(t + \tau)[\mathbf{A}(t + \tau, t)\mathbf{A}^T(t + \tau, t)]^{-1}\mathbf{y}(t + \tau) = \epsilon^2. \quad (7)$$

Because the mean square amplitude of the errors at time $t + \tau$ is the mean square length of the semiaxes of this ellipsoid, the Lorenz index $\alpha(t, \tau)$ is defined as

$$\alpha(t, \tau) = \left[\frac{1}{n} \sum_{i=1}^n \Gamma_i(t + \tau, t) \right]^{1/2}, \quad (8)$$

where Γ_i is the i th eigenvalue of the matrix $\mathbf{A}(t + \tau, t)\mathbf{A}^T(t + \tau, t)$. The square length of the i th semiaxis of the ellipsoid is given by $\epsilon^2\Gamma_i$.

Note that $\sum_i [\log \Gamma_i(t + \tau, t)] / (2\tau)$ corresponds to the volume contraction rate ($\sum_i \dot{x}_i / x_i$) in phase space, and has an invariant value. In the limit of $\tau \rightarrow 0$, the Lorenz index is well approximated by $[1 + 2(\tau/n) \times \text{trace}(\mathbf{J})]^{1/2}$; thus, it depends only on the diagonal components of \mathbf{J} at time t and does not represent instability characteristics of the motion. On the other hand, when $\tau \rightarrow \infty$, the Lyapunov exponents λ_i are obtained as $\lambda_i = \lim_{\tau \rightarrow \infty} (\log \Gamma_i) / (2\tau)$. The Lorenz index in this limit represents global characteristics of the predictability of the attractor.

It is clear from the definition that the Lorenz index gives the ensemble average of the linear growth rate of infinitesimally small initial errors. Therefore, when the error attains a finite amplitude with the increase of τ , the Lorenz index does not represent the real growth of the error for nonlinear effects. In this paper, however, we adopt the Lorenz index as a measure of the error growth because we aim at examining how the local error growth depends on the dynamics of the flow fields. This index does not depend on the amplitude nor the configuration of initial errors, but directly represents instability characteristics of the flow. On the contrary, if a nonlinear measure of the error growth, such as the spread among adjacent trajectories, is adopted, the relation between the error growth and the dynamics of the flow field may be obscured because any nonlinear measure of the error growth inevitably depends on the amplitude and configuration of initial errors. We also remark that the time interval τ of the Lorenz index should also be appropriately chosen so as to describe the local characteristics of the attractor. We compute the index for several values of τ in each case to assess the choice of the parameter value.

3. Results

We consider the Lorenz system (Lorenz 1963) with a set of ordinary differential equations:

$$\dot{X} = -\sigma X + \sigma Y, \quad (9)$$

$$\dot{Y} = -XZ + rX - Y, \quad (10)$$

$$\dot{Z} = XY - bZ, \quad (11)$$

where σ , r , and b are control parameters. An excellent review on dynamical behavior in the Lorenz system is given by Sparrow (1982). In this section, we investigate the motion in the intermittent chaos regime with $\sigma = 10$ and $b = 8/3$.

Pomeau and Manneville (1980) found an intermittent route to chaos in the Lorenz system around a large value of r . Below the critical value of $r_T (= 166.06)$, the system shows stable periodic variations. As r becomes slightly larger than r_T (Fig. 1), the fluctuation remains almost periodic during a long time interval (*laminar phase*), but the regular behavior is randomly and abruptly interrupted by a *burst*. The strange attractor generating this intermittency for $r = 166.2$ is depicted in Fig. 2 by a three-dimensional perspective, in which the trajectory for $10 \leq t \leq 100$ in Fig. 1 is drawn by a continuous line. The locations of two of the three unstable stationary points are also indicated by dots. In this figure, the laminar phase corresponds to a simply connected thick curve composed of accumulated trajectories in the attractor.

This intermittent chaos can be understood intuitively by constructing a one-dimensional Poincaré map as in Pomeau and Manneville (1980). Here, we take a Poincaré section of $-50 \leq X \leq 0$, $-100 \leq Y \leq -50$, and $Z = 120$, which is indicated by the shaded plane in Fig. 2. After recording points of the intersection (X , Y) of trajectories with this plane, we construct a one-dimensional map:

$$x_{n+1} = f(x_n), \quad (12)$$

where $x_n = (X_n^2 + Y_n^2)^{1/2}$ is the distance of the n th intersection from the Z -axis. Figure 3 shows the iterative map by small dots; a channel between the map and the bisector leads to the laminar phase because numerous iterations are necessary for the trajectory to travel through this narrow channel. Thus, the laminar

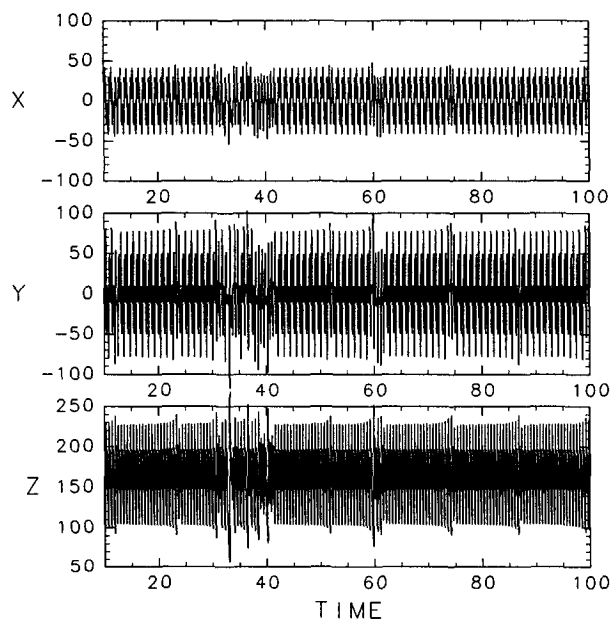


FIG. 1. Time variation of three variables (X , Y , Z) of the Lorenz system for $r = 166.2$, $\sigma = 10$, and $b = 8/3$.

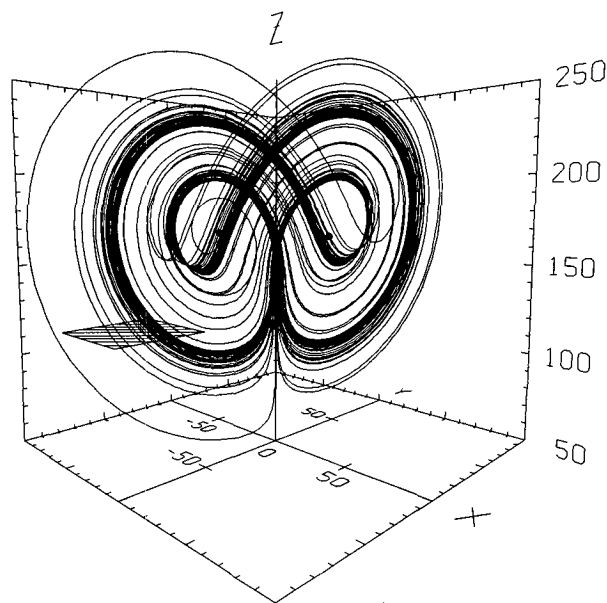


FIG. 2. Trajectory for Fig. 1 in a three-dimensional perspective. The locations of two of the three unstable stationary points are indicated by dots. A Poincaré section of $-50 \leq X \leq 0$, $-100 \leq Y \leq -50$, and $Z = 120$ is also denoted by the shaded plane.

phase corresponds to the quasi-stationary state in the one-dimensional map.

The temporal variation of the local error growth rate is computed in this intermittent chaos. Figure 4 shows the time sequence of the Lorenz index of Eq. (8) for $\tau = 0.1$. The time interval of 0.1 is very short compared with the time taken for the one return in the Poincaré section in Fig. 2, but long enough to represent instability characteristics of the motion. We had computed the index for some other values of τ (0.2, 0.3, and 0.4) and obtained qualitatively similar results. The temporal variation of the Lorenz index has two distinct time scales: one has a long time scale that corresponds to the transition between the laminar phase and the burst event; the other has a short time scale that reflects fine structures of the attractor. In this section, we examine characteristic variations of the Lorenz index with the long time scale. The variation with the short time scale will be examined in the next section in connection with the role of the unstable stationary point in organizing the local predictability.

In order to extract the long time scale variation, we calculate the Lorenz index for a time interval τ during which the trajectory makes one return in the Poincaré section. Note that τ depends on the number n of the intersection, but is nearly constant during the laminar phase because of the predominant periodicity. The solid line in Fig. 5 shows the sequence of the Lorenz index $\alpha(l_n, \tau_n)$. The abscissa denotes the number of the intersection n from $t = 10$ in Fig. 1. During the bursts, this index is not useful for the value of τ_n varying too

much following the iterations. (These portions of the curve are out of the frame in this figure.) We introduce a transiency index in the Poincaré map defined by $\|x_{n+1} - x_n\|$ to characterize the local dynamics. As indicated by the broken line in Fig. 5, the transiency index has a small value during the laminar phase, while it attains a very large value at the onset and the break of the laminar phase. On the other hand, the Lorenz index has a minimum value at the onset of the laminar phase, gradually (almost linearly with time) increases during the laminar phase, and attains a large value at the break of the laminar phase. The minima of the Lorenz index always precede the minima of the transiency index in time.

This characteristic relation between the Lorenz index and the transiency index is easily understood by considering the error growth in the one-dimensional map of Eq. (12). An infinitesimally small error y_n in the one-dimensional map is described by the linearized map:

$$y_{n+1} = \frac{d}{dx_n} f(x_n) y_n. \tag{13}$$

Therefore, the Lorenz index $\alpha(x_n, 1)$ for one iteration is written as

$$\alpha(x_n, 1) = \left\| \frac{d}{dx_n} f(x_n) \right\|. \tag{14}$$

During the laminar phase, the Poincaré map is well approximated by the following quadratic map (Pomeau and Manneville 1980):

$$x_{n+1} = f(x_n) = ax_n^2 + bx_n + c. \tag{15}$$

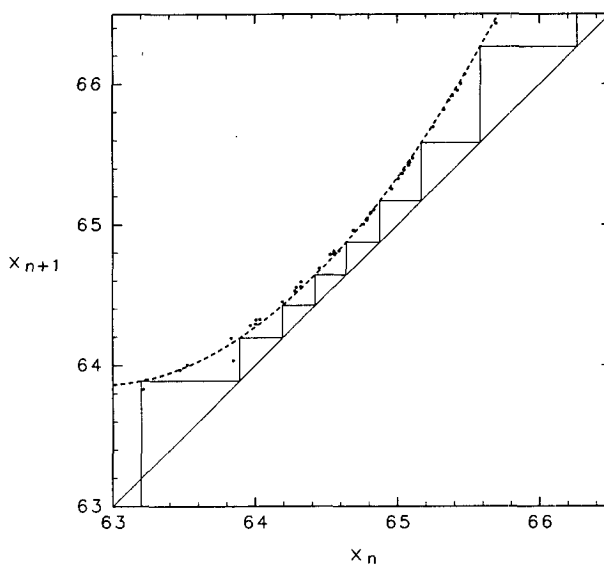


FIG. 3. A part of the Poincaré map along the distance from the Z-axis x_n for $r = 166.2$ (small dots). Broken curve is computed using a least squares method by $x_{n+1} = ax_n^2 + bx_n + c$.

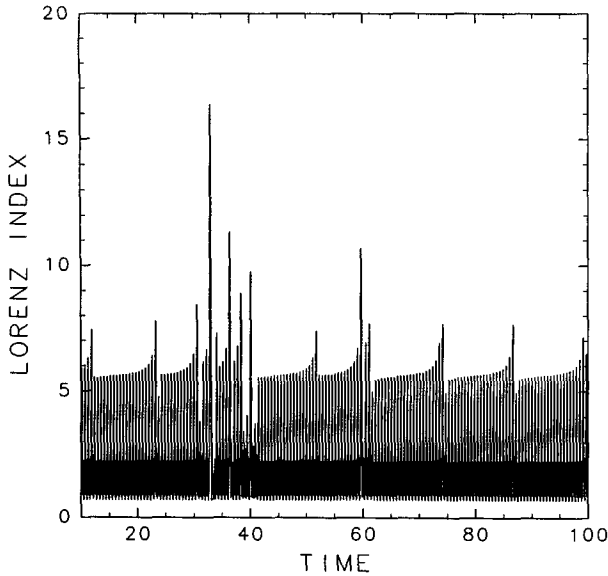


FIG. 4. Time variation of the Lorenz index $\alpha(t, 0.1)$ for $r = 166.2$.

The broken curve in Fig. 3 denotes this map in which the parameters a , b , and c are obtained by a least squares method. The Lorenz index $\alpha(x_n, 1)$ for this map is given by

$$\alpha(x_n, 1) = \|2ax_n + b\|. \quad (16)$$

Figure 6 shows the variation of the transiency index and that of the Lorenz index for the map of Eq. (15) near the channel region. The transiency index attains its minimum value at $x_n = x_c$, which is the nonstationary local minimum point in this one-dimensional map (see Mukougawa 1988). On the other hand, the Lorenz index increases linearly with x_n and equals unity at $x_n = x_c$. Thus, the initial error decreases when the iteration approaches the local minimum point. On the contrary, it increases when the iteration departs from the local minimum point. This relation between the Lorenz index and the transiency index in the one-dimensional map well represents that in Fig. 5 except for the value of the Lorenz index; the Lorenz index in Fig. 5 is always greater than unity. This discrepancy arises from the restriction of the motion in the one-dimensional map in Fig. 6.

The long time scale variation of the two indices during the laminar phase in the three-dimensional Lorenz system is, therefore, attributed to the local property of the local minimum point in the one-dimensional map.

4. Discussion

Intermittent chaos examined in this study is characterized by an inverse tangent bifurcation in which two stationary points merge in the Poincaré map, classified as type I intermittency by Pomeau and Manneville (1980). In the framework of the one-dimensional

map, the unstable stationary point can also generate intermittent chaos referred to as type III. In the vicinity of the unstable stationary point, the iteration is well approximated by a linear map in contrast with the quadratic map in type I intermittency. As a result, the Lorenz index has an almost constant value during the laminar phase of type III intermittency. It has no relation with the transiency index as in type I intermittency. Therefore, the temporal variation of the local error growth rate crucially depends on the type of the intermittency.

Recently, Nese (1989) also investigated the temporal variation of the local divergence rate of adjacent trajectories and its phase-spatial organization on the Lorenz attractor. He used the largest "Lyapunov exponent" defined for a finite time interval as a measure of the local error growth. Because his index depends on the configuration of the initial error, the local divergence rate does not have a unique value at any point on the attractor [see Fig. 4 and Eq. (4) in his paper], but exhibits large fluctuations. On the contrary, the Lorenz index indicating the ensemble average of the error growth rate is a smooth function of the variables. Figure 7 shows phase-spatial distribution of the Lorenz index $\alpha(t, 0.1)$ on the *standard* Lorenz attractor for $r = 28.0$. We can obtain the organization of the local predictability without any averaging procedure in contrast with Nese (1989). If we use his index as a local predictability, we cannot detect the characteristic relation between the local predictability and the local dynamics precisely. This remark is also relevant to the investigation of the local error growth in a huge dynamical system, such as operational forecast models. In these systems, the time evolution can be calculated

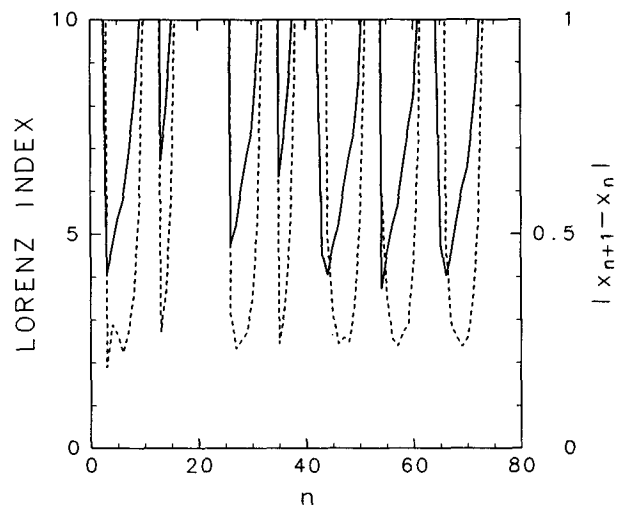


FIG. 5. Iterations of the Lorenz index $\alpha(t_n, \tau_n)$ for a time interval τ_n during which the trajectory makes one return in the Poincaré section (solid line) and the transiency index of $\|x_{n+1} - x_n\|$ (broken line) during the laminar phase. The number of the intersection is recorded from $t = 10$ in Fig. 1.

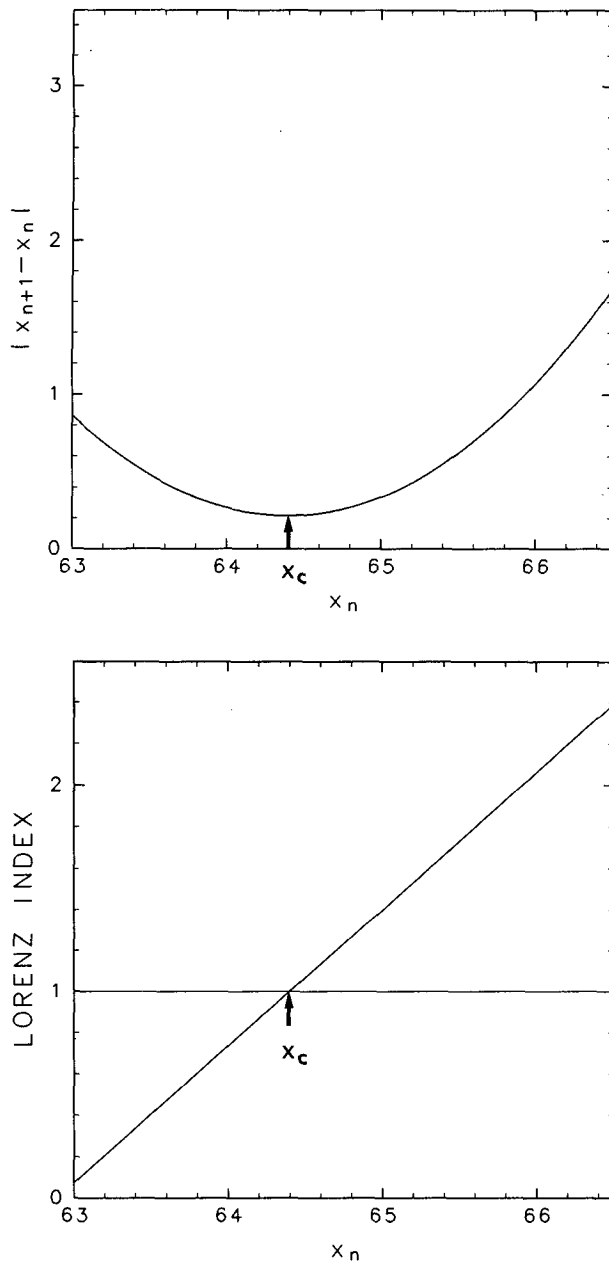


FIG. 6. The variation of $\|x_{n+1} - x_n\|$ (a) and the Lorenz index $\alpha(x_n, 1)$ in the one-dimensional map (b) for the quadratic map of Eq. (15). The local minimum point is also denoted by x_c .

only for a limited number of the configuration of initial errors. Indeed, even in the conventional *ensemble* forecasts, such as the Monte Carlo forecast (Leith 1974) and the lagged average forecast (Hoffman and Kalnay 1983), the number of initial states is no more than ten. Thus, the deduced measure of the local predictability may not correspond to the *true* local predictability as in Nese's results. We should keep this defect in mind in interpreting the result.

Legras and Ghil (1985) suggested the relationship between the temporal variation of the predictability and the dynamics of quasi-stationary states by using a barotropic model with 25 variables. At the onset of quasi-stationary states, the rate of divergence of nearby trajectories, which was measured by the largest eigenvalue of the linearized instantaneous field, has a small value. On the contrary, it has a large value at the break. They tried to explain this variation of the predictability by the local dynamics of the unstable stationary point generating quasi-stationary states: Persistencies are associated with gradual capture of the trajectory into a contracting phase flow region near the stable manifold of the unstable stationary point and rapid transients with strong instabilities along the unstable manifold. We computed the Lorenz index along the stable and unstable manifold of the three unstable stationary points in the Lorenz system. However, none of the unstable stationary point has such a property suggested by them. This is also implied by the phase-spatial organization of the Lorenz index in Fig. 7; the Lorenz index does not show any distinctive organization (e.g., a local maximum or minimum value) around these unstable stationary points denoted by dots in this figure. Therefore, all unstable stationary points do not have the same local dynamics.

Palmer and Tibaldi (1988) and Chen (1989) suggested by a statistical approach with operational forecast models that the persistence of the model forecast can be employed as the predictor of the forecast skill. However, if we consider that the persistency is caused by the unstable stationary points (and also nonstation-

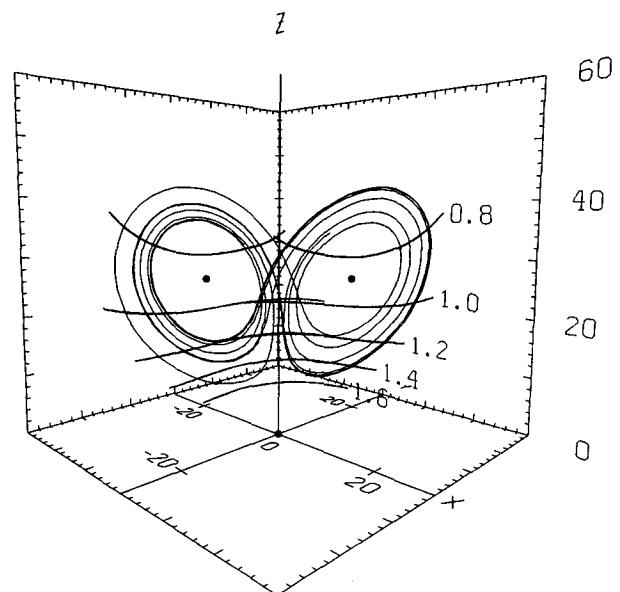


FIG. 7. Distribution of the Lorenz index $\alpha(t, 0.1)$ on the Lorenz attractor for $r = 28.0$ in a three-dimensional perspective. A part of the trajectory is indicated by the solid line. The locations of three unstable stationary points are also denoted by dots.

ary local minimum points), we can say that such a simple statistical approach has a limitation because the local minimum point may have different local dynamics and different properties of the local error growth. Thus, it is more desirable to investigate the characteristics of the local predictability for each weather regime separately.

Use of the Lorenz index to examine whether the relation between the local predictability and the transiency exists is practically impossible in huge dynamical systems, such as the operational forecast models with $O(10^6)$ variables. The present computer resources cannot handle such a huge error matrix of $O(10^6)$ by which the Lorenz index is evaluated. This formidable task to examine the *complete* Lorenz index corresponding to the full system of the forecast model is also physically meaningless because many components of small scales irrelevant to the medium-, or extended-range forecasts constitute the largest part of the error matrix. Therefore, we must exert ourselves to extract large-scale components that characterize the variation of local predictability, and to evaluate the reduced Lorenz index corresponding to the extracted components. This effort corresponds to the procedure to take the Poincaré map in this study, by which the transversal properties of chaotic motions can be clearly captured. For future work, we will attempt to examine the validity of this *reduced* Lorenz index in forecasting the forecast skill of the operational forecast models.

5. Conclusion

A role of the quasi-stationary state in determining the phase-spatial organization of the local error growth rate was investigated in the intermittent chaos regime of the Lorenz system. As a measure of the local error growth rate, we adopted the Lorenz index, which gives the amplification rate of the rms error during a prescribed time interval. The following results were obtained:

- In the intermittent chaos, the laminar phase is an almost periodic state. The Lorenz index for the one return in the Poincaré section has a minimum value at the onset of the laminar phase, gradually increases during the laminar phase, and attains a large value at the break of the laminar phase.
- This characteristic evolution of the Lorenz index is well explained by considering the local error growth in the one-dimensional Poincaré map, in which the laminar phase is characterized by a local minimum point (i.e., a quasi-stationary state). The Lorenz index in the one-dimensional map is smaller than unity when the iteration approaches the local minimum point. On the contrary, it is larger than unity when the iteration departs from the local minimum point.

- The local dynamics of the unstable stationary point in the Lorenz system does not affect fine phase-spatial organizations of the Lorenz index on the attractor.

For a further study, we must investigate the relationship between the temporal variation of the predictability and the dynamics of the quasi-stationary states in huge dynamical systems. In particular, the applicability of the one-dimensional picture of the local organization of the predictability by the local minimum point must be examined in detail. Our final goal is, of course, to develop the method to predict the prediction skill in the numerical weather forecast by utilizing this kind of new concept of the quasi-stationary states.

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