CLASSIFICATION OF SIMPLE LOW-ORDER MODELS IN GEOPHYSICAL FLUID DYNAMICS AND CLIMATE DYNAMICS

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1. INTRODUCTION

Lorenz derived the famous three-variable dynamical system [1] that contains chaotic solutions in some parameter ranges from a numerical model on 2-dimensional Rayleigh-Bénard convection. He called this kind of highly simplified system low-order model. In the fields of Geophysical Fluid Dynamics (GFD) and Climate Dynamics (CD), many descendants of the low-order model have been developed to study the essential dynamics of the original much complex systems.

In this review article, these low-order models are classified into several groups with the knowledge of nonlinear dynamical systems. Zero-dimensional energy balance climate model and Lorenz's "general circulation model" are used as examples of each group. The grouping is as follows: (I) multiple equilibria in nonlinear systems, (II) periodic and chaotic solutions in autonomous systems, and (III) periodic and chaotic solutions in nonautonomous systems.

2. MULTIPLE EQUILIBRIA IN NONLINEAR SYSTEMS

A zero-dimensional energy balance model (0-d EBM) [2,3], which is the simplest conceptual model of the earth's climate, is analyzed as an example of nonlinear low-order models with multiple equilibria.

The 0-d EBM is a dynamical system with a dependent variable \( T \) of a mean temperature of the earth-atmosphere system:

\[
C \frac{dT}{dt} = R_s(T) - R_o(T),
\]

where \( t \) is time, \( C \) effective heat capacity of the system, \( R_s(T) \) net solar radiation absorbed by the system, and \( R_o(T) \) longwave radiation emitted to space. The net absorption of the solar radiation is a function of \( T \) through the planetary albedo \( \alpha(T) \):

\[
R_s(T) = \frac{S_0(1 - \alpha(T))}{4},
\]

where \( S_0 \) is the solar constant (= 1.37 \( \times 10^3 \) Wm\(^{-2}\)). Here the planetary albedo is simply assumed to be a piecewise linear function of \( T \) due to the presence of ice on the earth:

\[
\alpha(T) = \begin{cases} 
0.3, & 270K \leq T, \\
0.7 + (0.3 - 0.7) \times \frac{T - 230}{270 - 230}, & 230K \leq T < 270K, \\
0.7, & T < 230K.
\end{cases}
\]

The longwave radiation emitted to space basically obeys Stefan-Boltzmann's law of radiation and it is linearized as follows,

\[
R_o(T) = A + BT,
\]
where coefficients are estimated from satellite observations as $A = -363\text{Wm}^{-2}$ and $B = 2.1\text{Wm}^{-2}\text{K}^{-1}$ [4].

Steady solutions $\bar{T}$ of the system (2.1) are obtained by setting the l.h.s. equal to zero. For the parameter values given above, there are three steady solutions of $\bar{T} = 287\text{K}$, $243\text{K}$ and $222\text{K}$ (denoted by $\odot$, $\ominus$ and $\bigcirc$, respectively) because of the nonlinearity of $a(T)$.

Linear stability of the steady solutions is studied by introducing a small perturbation from each steady solution $T' = T - \bar{T}$. Eigenvalue of the Jacobian matrix, which is simply a derivative of the r.h.s. of Eq.(2.1) in this one-variable model, gives growth rate of the perturbation; the steady solutions $\odot$ and $\ominus$ are stable, while $\bigcirc$ is unstable. Therefore, multiplicity of stable steady solutions exists for the present parameter values.

As for the nonlinear stability, a potential function $P(T)$ can be constructed and Eq.(2.1) is written as

$$C \frac{dT}{dt} = -\frac{dP}{dT},$$

$$P(T) = -\int_0^T \left( R(T') - R(\bar{T}) \right) dT'.$$

If $T$ is larger than the value that gives the local maximum of $P(T)$, which is the unstable steady solution $\bigcirc$, then the time-dependent solution monotonically converges to the local minimum of $P(T)$ for the stable steady solution $\odot$. If $T$ is smaller than that value, on the other hand, it converges to $\ominus$.

In the segments that contain the stable steady solutions, relaxation to each stable steady solution is given by

$$C \frac{dT}{dt} = -BT + \beta_i, \quad (i = 1 \text{ or } 3),$$

because $\alpha$ is independent of $T$. Therefore the time-constant for the relaxation is given by $\tau \equiv C/B$. The heat capacity $C$ of the atmosphere is about $10^7\text{JK}^{-1}\text{m}^{-2}$ and $\tau \sim 55$ days. If the mixing layer of the ocean is included into the system ($C \sim 10^8\text{JK}^{-1}\text{m}^{-2}$), $\tau$ becomes 1.5 years; while it becomes 150 years for the whole ocean ($C \sim 10^{10}\text{JK}^{-1}\text{m}^{-2}$). Generally ice-sheet models use the relaxation time of the order of $10^4$ years.

The multiplicity of steady solutions is dependent on the external parameters. If the variation of solar luminosity (irradiance) is taken into account, the solar radiation is replaced by $\mu S_\odot (\mu = 1$ for the present condition). Figure 1 is a bifurcation diagram of steady solutions with the control parameter $\mu$. There are two limit points, $L_1$ and $L_2$, in the diagram, between which multiplicity of stable steady solutions ($\odot$ and $\bigcirc$) is possible for $0.84 < \mu < 1.17$. 

![Figure 1](Image)
Table 1 Examples of multiple stable states in GFD and CD.

<table>
<thead>
<tr>
<th>item</th>
<th>experiment/observation/review</th>
<th>low-order models</th>
</tr>
</thead>
<tbody>
<tr>
<td>atmospheric blocking</td>
<td>Benzi et al. (1986) [12]</td>
<td>Charney and DeVore (1979) [13,7]</td>
</tr>
</tbody>
</table>

Above result in the 0-d EBM shows a possibility of two largely different stable climates for the present solar condition. This possibility was firstly pointed out by the use of one-dimensional EBMs [5,6], in which the dependent variable $T$ is a function of time and latitude and the effect of the meridional heat transport is incorporated. Bifurcation diagram for the 1-d EBM shown in [7] is qualitatively similar to Fig.1. Two stable states of the present climate and an ice-covered white earth for the same solar radiation were also obtained in the time-integrations of a much sophisticated general circulation model (GCM) with different initial conditions [8].

"Multiple equilibria" is an interesting paradigm in GFD and CD as listed in Table 1 because there are several observations which are suggestive of the existence of more than one stable states for the same external conditions. Some laboratory experiments on thermal convection show stepwise transitions from regular flow to chaotic one, which are interrupted as successive bifurcations. Hysteresis, which is an indication of multiple stable states, is often observed around each transition. Bimodality of the flow regimes has been observed in the atmosphere and ocean; straight or meander path of the jet stream in the midlatitude troposphere (the meander is known as the blocking), strong persistent polar vortex or weak disturbed one in the winter stratosphere, straight or meander path of Kuroshio, the western boundary current of the Pacific ocean. Moreover, multiple stable equilibria may explain not only the duality of the earth’s climate but also that of the atmospheric circulation of the Venus.

3. PERIODIC AND CHAOTIC SOLUTIONS IN AUTONOMOUS SYSTEMS
— DELAYED OSCILLATOR —

Periodic and chaotic solutions may exist in autonomous systems, in which external parameters are constant with time. They are possible even in one-variable model if the system has a term with time-delay. A first-order delay differential equation over the time interval $[-\delta, 0]$ corresponds to an infinite-dimensional dynamical system without delay.

An example of such systems with delay is a conceptual model of internal climate variation [7], that is, 0-d EBM with delay and kink [20]. Here we assume a delay $\delta$ in the icd-albedo feedback; the albedo $\alpha$ is determined by the temperature at the time before $\delta$:

$$C \frac{dT(t)}{dt} = \frac{S_0}{4} \{1 - \alpha(T(t - \delta))\} - \{A + BT(t)\}. \tag{3.1}$$

Furthermore we introduce a kink of $\alpha(T)$ around $T = 280$ K, because it is necessary to obtain nonstationary solutions in Eq.(3.1). Equation (2.3) is modified with the following kink:

$$\alpha(T) = \begin{cases} 
0.3 + (d - 0.3) \times \frac{T}{280} & 275K \leq T < 280K, \\
\frac{T}{280 - 275} & 280K \leq T < 285K, \\
d + (0.3 - d) \times \frac{T - 280}{285 - 280} & 285K \leq T < 285K.
\end{cases} \tag{3.2}$$
Fig. 2. Periodic and chaotic solutions of the zero-dimensional energy balance model with delay and kink. 
\( \delta = 92 \) years (a), 112 years (b), 150 years (c) and 500 years (d).

This kink is a crude representation of the effect of cloudiness in baroclinic zone [21,20].

Steady solutions of Eq. (3.1) are obtained in the same way as the system without delay. Two steady solutions of \( T = 284K \) (a) and 277K (b) are added to (c) due to the existence of kink \( \delta = 0.4 \). If there is no delay, the steady solution (a) is stable while (b) is unstable. Linear stability in the case with delay is analyzed by introducing a perturbation of the following form, \( T' = \theta_0 e^{\lambda t} \). The growth rate \( \lambda \), which is complex in general, is given by the following algebraic-exponential equation:

\[
\lambda = -\frac{1}{C} \left\{ \frac{S_0}{4} \left( \frac{d\alpha}{dT} \right) \right\} e^{-\lambda \delta} + B = -\bar{a}e^{-\lambda \delta} - \delta.
\]

(3.3)

The steady solutions (a) and (b) are always stable for any \( \delta \) because \( \bar{a} = 0 \) and \( \delta > 0 \). An analysis of Eq. (3.3) [20] shows that the steady solutions (c) and (d) are always unstable for any \( \delta \). The steady solution (e) changes the stability at a critical value,

\[
\delta_c = \left\{ \pi - \arccos(\bar{a}/\delta) \right\} / \text{Im}[\lambda_c],
\]

(3.4)

where \( \text{Im}[\lambda_c] = \sqrt{\bar{a}^2 - \delta^2} \). It is unstable for the delay longer than this critical value, and the period of the oscillation at the critical value is \( 2\pi/\text{Im}[\lambda_c] \).

Asymptotic nonstationary solutions of Eq. (3.1) are obtained by numerical time integrations. Figure 2 shows some examples of periodic and chaotic solutions for the whole atmosphere-ocean system with \( C = 10^{10}JK^{-1}\text{m}^{-2} \), for which \( \delta_c = 91.5 \) years. Near the critical value the asymptotic solution is a periodic solution.

**Table 2: Examples of delayed oscillator in GFD and CD.**

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<thead>
<tr>
<th>item</th>
<th>textbook</th>
<th>delayed-oscillator models</th>
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<tbody>
<tr>
<td>glaciation cycles due to ice-albedo feedback</td>
<td>[4,7]</td>
<td>[20,21]</td>
</tr>
<tr>
<td>ENSO</td>
<td>Philander(1990) [22]</td>
<td>Suarez and Schopf(1988) [23]</td>
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<td></td>
<td></td>
<td>Battisti and Hirst(1989) [24]</td>
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</table>
close to the linear analysis with a period of 309 years (a). If the delay $\delta$ is increased, a period-doubling bifurcation takes place (b,c) and finally chaotic solutions appear (d). Note that the dominant period of the chaotic solution (d) is 1310 years and much longer than the relaxation time of 150 years.

There are some delayed oscillator models in GFD and CD as listed in Table 2. In addition to the 0-d [20] and 1-d [21] EBM$^*$s with delay and kink, some delayed oscillator models [23,24] were constructed to illustrate the periodic occurrence of El Niño and the Southern Oscillation (ENSO). The delay is due to oceanic waves propagating in equatorial zone.

4. PERIODIC AND CHAOTIC SOLUTIONS IN AUTONOMOUS SYSTEMS — MULTI-DIMENSIONAL SYSTEMS —

Another type of the conceptual model of internal climate variation is described with Lorenz's "GCM" [26]. Periodic solutions may exist in autonomous systems even without time-delay if the system has two or more dependent variables, and chaotic solutions in systems with three or more dependent variables. The Lorenz model is a 3-variable nonlinear autonomous system;

$$\begin{align*}
\frac{dX}{dt} &= Y^2 - Z^2 - aX + af, \\
\frac{dY}{dt} &= XY - bXZ - Y + g, \\
\frac{dZ}{dt} &= bXY + XZ - Z,
\end{align*}$$

(4.1)

where $t$ is time, $X(t)$ intensity of westerly zonal flow, $Y(t)$ and $Z(t)$ cos and sin components of a planetary wave, respectively. Four external parameters are $a$: damping factor for the zonal flow, $b$: phase propagation speed of the planetary wave, $f$: forcing of the zonal flow, and $g$: forcing of the planetary wave. This is a forced-dissipative system with energy source and sink.

Firstly, a special case without wave forcing is analysed by setting $g = 0$. In this case a steady solution without wave components is possible;

$$X = f, \quad Y = Z = 0.$$  

(4.2)

This kind of zonally symmetric steady solution is called as Hadley solution. Linear stability of the Hadley solution is given by the Jacobian matrix of the r.h.s. of Eqs.(4.1). Because the eigenvalues of the matrix are $\lambda = -a$ and $\lambda = f - 1 \pm ib|f|$, the Hadley solution is stable for $f < 1$ and unstable for $f > 1$. At the critical point $f = 1$ the latter eigenvalues are pure imaginary and Hopf bifurcation takes place. The following periodic solution exists for $f > 1$:

$$\begin{align*}
X &= 1, \\
Y &= \sqrt{a(f - 1)} \cos b(t - t_0), \\
Z &= \sqrt{a(f - 1)} \sin b(t - t_0).
\end{align*}$$

(4.3)

This solution shows a constant eastward migration of the wave for $b > 0$ without interactions with the zonal flow.

If the wave forcing $g$ exists in the system, on the other hand, the Hadley solution does not exist. Steady solutions are obtained by setting $d/dt = 0$ in Eqs.(4.1):

$$a(X - f)(1 + b^2)X^2 - 2X + 1) + g' = 0.$$  

(4.4)

Because $X \in \mathbb{R}$, one or three steady solutions are possible depending on the external parameters $a$, $b$, $f$ and $g$. If $X$ is determined, wave components are given by

$$\begin{align*}
Y &= g(1 - X)/(1 + b^2)X^2 - 2X + 1), \\
Z &= bgX/(1 + b^2)X^2 - 2X + 1).
\end{align*}$$

(4.5)

As shown in Fig.3, three steady solutions exist between two limit points, $L_1$ and $L_2$, because of the nonlinear resonance of forced planetary waves as in [13]. However, multiple stable equilibria is limited in the narrow range around $f = 1.2$. 
Fig. 3. Bifurcation diagram of Lorenz’s GCM showing the dependence of the intensity of westerly zonal flow $X$ on the external parameter $f$ of the zonal flow forcing with $a = 0.25$, $b = 4$, and $g = 1$. Stable steady solution is denoted by $+$, unstable ones by $+$ (real and positive eigenvalue) and $\times$ (complex eigenvalues with positive real part). Local maximum or minimum of nonstationary solutions is also shown by $\cdot$. If the number of this small dot is finite for a given $f$, it is a periodic solution.

A Hopf bifurcation takes place at $f \approx 1.28$ and stable periodic solutions are obtained for a wide range of $f$ as denoted by a couple of small dots in Fig. 3. Multiplicity of stable solutions (steady and periodic solutions) exists in this wide parameter range. A period-doubling bifurcation takes place around $f \approx 6.2$ and chaotic solutions are obtained for $7.6 < f < 8.8$. Another type of periodic solution exists stably for $8.8 < f$.

Many of the low-order models listed in Table 1 for examples of multiple stable states in GFD and CD have periodic and chaotic solutions in some parameter ranges. Even if the most simplified system such as [13] has only steady solutions as attractor of the system, periodic and chaotic solutions may be obtained in the systems with the same dynamical framework but with more degrees of freedom (in other words, such low-order models have sensitivity to the way of simplification). Some other examples of periodic and chaotic solutions in low-order models in GFD and CD are listed in Table 3. Low-order models [28,31,32] have only two dependent variables, which is enough to have periodic solutions, and models [29,30] have three variables to have chaotic solutions.

<table>
<thead>
<tr>
<th>item</th>
<th>textbook</th>
<th>low-order models</th>
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<tbody>
<tr>
<td>chaotic model of ENSO</td>
<td>[22]</td>
<td>Vallis (1986, 1988) [29, 30]</td>
</tr>
<tr>
<td>glaciation cycles</td>
<td>[4, 7]</td>
<td>Källén, Crafoord and Ghil (1979) [31]</td>
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<tr>
<td></td>
<td></td>
<td>Saltzman and Moritz (1980) [32]</td>
</tr>
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</table>
5. PERIODIC AND CHAOTIC SOLUTIONS IN NONAUTONOMOUS SYSTEMS

Nonstationary solutions in nonautonomous systems, in which external parameters have time variations, are subdivided into three groups by the characteristic of each system that is obtained when the external parameters are constant with time. (III-1) linear system, (III-2) nonlinear system with only steady solutions if there is no time-dependent forcing, (III-3) nonlinear system with nonstationary solutions even if there is no time-dependent forcing.

Firstly, response of a linear system to a periodic forcing (III-1) is analyzed. As an example, the 0-d EBM for a piecewise linear system (2.7) is used by introducing a periodic forcing term with an amplitude $\gamma$ and a frequency $\omega_f$:

$$C \frac{dT}{dt} = -BT + \beta + \gamma e^{i\omega_f t}. \quad (5.1)$$

If $\gamma = 0$, Eq.(5.1) has a stable steady solution $T = \beta / B$, because $B > 0$. Now, let $T'(t) = T(t) - T$, then Eq.(5.1) becomes

$$C \frac{dT'}{dt} = -BT' + \gamma e^{i\omega_f t}. \quad (5.2)$$

Assuming a form of $T' = \theta e^{i\omega_f t}$, and substituting this into (5.2), we obtain a solution

$$T' = \frac{\gamma}{R(1 + i\omega_f \tau)} e^{i\omega_f \tau} = \frac{\gamma}{R(1 + i\omega_f \tau)^{1/2}} e^{i(\omega_f \tau + \phi)}, \quad \phi = \arctan(-\omega_f \tau), \quad \tau = (C/B)$$

where $\tau = C/B$ is the time-constant for relaxation.

If the forcing period is much longer than $\tau$ (i.e., $\omega_f \tau \ll 1$), then Eq.(5.3) is approximated by $T' \sim \gamma B^{-1} e^{i\omega_f t}$; the amplitude of the response is proportional to that of the forcing and the phase lag becomes close to zero. Therefore quasistatic arguments are reasonable for this limit. If the forcing period is much shorter than $\tau$, on the other hand, Eq.(5.3) is approximated by $T' \sim \gamma (B\omega_f \tau)^{-1} e^{i(\omega_f \tau - \phi)}$. The response becomes small in proportion to $1/\omega_f$ and the phase lag approaches $\pi/2$.

In nonlinear systems, on the other hand, their response to a periodic forcing is various depending on the characteristics of each system. As an example of the group (III-2) of nonlinear systems, the 0-d EBM with a kink (3.2) of $d = 0.35$ but without delay is analyzed:

$$C \frac{dT}{dt} = \mu \frac{d}{4} \{1 - \alpha(T)\} - (A + BT), \quad (5.4)$$

where a periodic variation of the solar irradiance is incorporated as

$$\mu(t) = 1 + \delta \mu(t) = 1 + \gamma e^{i\omega_f t}. \quad (5.5)$$

Responses of the system around two stable solutions $T_1$ and $T_3$ are particularly interesting because two limit points exist even for a small variation of $\mu$ as shown in Fig.4(left). If $\gamma$ is so small that any limit point does not exist in the variable range of $\mu$, the response is basically the same as that in the linear system discussed above. When $\gamma$ is large enough ($\gamma > 0.02$) to have two limit points within the variable range, on the other hand, the response is very different depending on $\omega_f$, the ratio of the relaxation time to the forcing period. If the forcing period is much longer than $\tau$, transitions between $T_1$ and $T_3$ take place around the limit points and periodic responses with large amplitude are obtained as shown in Fig.4(a,b). This feature is known as hysteresis; the selection of the solution branch around $\delta \mu \sim 0$ depends on the past evolution. Time-series of the response of hysteresis (Fig.4 right) shows fast transitions even for $\tau \omega_f \sim 0.1$ (b). It is a periodic response with the same frequency of the forcing $\omega_f$ but higher harmonics $j \omega_f (j = 2, 3, 4, \cdots)$ also have large amplitude. If the forcing period is comparable to $\tau$ or shorter than that, on the other hand, there is no transition; the response is a weak periodic variation around either $T_1$ or $T_3$ depending on the initial condition as shown in Fig.4(c).
Fig. 4. Response of the system (5.4) to a periodic variation of $\mu$. (left) thick lines show periodic orbits in the $\delta \mu$-$T$ plane, and solid and dotted thin lines are stable and unstable steady solutions, respectively; (right) dotted line is the time evolution of $\delta \mu$, and thick solid and dashed lines are those of $T$. Nondimensionalized forcing frequency $\tau \omega_f$ is 0.01 (a), 0.1 (b) and 1 (c).

In the above analysis the external forcing is assumed to be monochromatic. If the forcing has two or more frequencies, the total response of a linear system is just a superposition of the response to each frequency in the form of Eq.(5.3). For two forcing frequencies $\omega_1$ and $\omega_2$, time-series of the response shows a beat of the frequency $|\omega_1 - \omega_2|$ but no spectral power for this frequency. Similar response is also obtained in nonlinear systems as shown in Fig.5(a,b) if the amplitudes of the forcing are small. When the forcing amplitudes are large enough to make intermittent transitions between two solution branches, on the other hand, the power spectra of the response have several peaks at $|j\omega_1 \pm k\omega_2|$ as well as $|\omega_1 - \omega_2|$. This kind of oscillations is known as “combination oscillations” [7,33].

Now, Lorenz’s “GCM” (4.1) is analyzed as an example of the nonlinear systems (III-3) which have nonstationary solutions even if there is no time-dependent forcing. An annual variation is introduced in the zonal flow forcing as in [34], $f = f + 2 \cos(2\pi t/365\text{days})$. Both periodic and chaotic solutions exist in this variation range of $f$ as seen in Fig.3. A time-series $X(t)$ of the response has large interannual variability of dual seasonal march in “summer” [34,35]. Therefore, a power spectrum for the time-series $X(t)$ (Fig.6) shows ultra-low-frequency variations with red-noise characters as well as sharp peaks of annual variations. Such ultra-low-frequency variations are possible even if the relaxation time ($\sim$ a week) is not so long, when the system has multiplicity of stable states in a part of the variation range. One of the states is chosen randomly in each year.

Another example of the nonlinear systems (III-3) with nonstationary solutions is a glaciation-cycle model with free internal oscillations [31]. Ghil and Le Treut [36] added a small amplitude of the external forcing with a period of $10^5$ year to the model with an internal oscillation of $6.7 \times 10^3$ year period, and obtained a large response at the forcing frequency. They called this large response “nonlinear resonance” in the analogy of the linear resonance, because the response at the forcing frequency is very weak when the internal oscillation is...
suppressed. In the nonlinear resonance, part of the internal variability is transferred to the forcing frequency.

Table 4 summarizes periodic forcings to the climate system of the earth. The solar cycle is not so exactly periodic as others, because it is an oscillation due to internal dynamics of the sun. There are many signals of periodic response to the daily and annual forcings. Irregular fluctuations generated internally in the complex system are also observed in the same frequency range at least up to the period of years. In the same way, irregular fluctuations in the time scale of glaciation cycles can be expected as well as the periodic response of Milankovitch cycles if the climate system is nonlinear.

Until now we have referred to only deterministic models. On the other hand, several stochastic climate models have been proposed since the work by Hasselmann [37]. In such models, slow changes of “climate” are described explicitly by deterministic equations while the effect of short-period “weather” disturbances is incorporated as random forcing terms. Such climate models act as an integrator of the short-period stochastic excitation.

Introducing a small-amplitude periodic forcing in a stochastic climate model, Benzi et al. [38] proposed a concept of “stochastic resonance”, which is now known to occur in a wide range of physical systems [39]. Stochastic resonance is a nonlinear cooperative effect in which stochastic term plays a constructive role in the

Table 4 Responses of the climate system of the earth to periodic forcings.

<table>
<thead>
<tr>
<th>item</th>
<th>periodic forcing</th>
<th>responses</th>
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<tbody>
<tr>
<td>daily variation</td>
<td>rotation of the earth</td>
<td>thermal tide, land and sea breeze,</td>
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<td></td>
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<td>mountain and valley breeze</td>
</tr>
<tr>
<td>annual variation</td>
<td>revolution of the earth</td>
<td>seasonal variations, Monsoon</td>
</tr>
<tr>
<td>11-year solar cycle</td>
<td>internal variation of the sun</td>
<td>upper atmosphere,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>middle or lower atmosphere(?)</td>
</tr>
<tr>
<td>orbital variations</td>
<td>obliquity, precession, eccentricity</td>
<td>Milankovitch cycles</td>
</tr>
</tbody>
</table>
detection of weak periodic forcings. We consider the 0-d EBM (2.5) with a stochastic term,

\[
\frac{dT}{dt} = \frac{1}{C} \frac{dP}{dT} + \epsilon \eta(t),
\]

where \( \eta(t) \) is a normalized Wiener process (white noise) and \( \epsilon \) is the amplitude of the stochastic process, and assume a periodic variation of the potential function \( P \) that has two stable states of \( 1 \) and \( 2 \) similar to the system used for Fig. 4. When the variation of \( P \) is small, the response is also small for \( \epsilon = 0 \), because there is no transition. For a finite value of \( \epsilon \), however, a large response at the forcing period of \( P \) is obtained due to the transitions between \( 1 \) and \( 2 \). For much larger \( \epsilon \) the small periodic forcing is not detectable again by a large stochastic response. Thus the stochastic resonance is an example of a periodic response of nonlinear systems with both external (small) periodic forcing and internal stochastic variations.

6. CONCLUSIONS

The knowledge of nonlinear dynamical systems is useful to classify simple low-order models developed in the fields of Geophysical Fluid Dynamics (GFD) and Climate Dynamics (CD). In this review such low-order models are classified into three groups (I) conceptual models of climate, (II) conceptual models of internal climate variation, and (III) conceptual models of climate variation due to external periodic forcings.

Group (I) is characterized by multiple stable states in nonlinear autonomous systems. There are many examples of low-order models on energy balance of the earth’s climate system, thermal convection, sloping convection, meander of jet stream, bimodality of stratospheric winter circulation, meander of KUROSHIO, four-day circulation in the Venus atmosphere, and so on.

Group (II) is subdivided into two: nonlinear autonomous systems with delay and multi-dimensional nonlinear autonomous systems. Examples of the former are glaciation cycles in energy balance models and El Niño Southern Oscillation (ENSO) models, while those of the latter are Lorenz’s “General Circulation Model (GCM)”, meander of jet stream, intraseasonal variations of stratospheric circulation, Quasi-Biennial Oscillation in the equatorial stratosphere, ENSO, and glaciation cycles.

Nonlinear systems of Group (III) are subdivided into two by the condition whether the system has steady solutions or nonstationary solutions when the forcing is constant with time. Some glaciation-cycle models with only steady solutions show hysteresis and combination oscillation, while those with internal periodic or
stochastic process show nonlinear resonance or stochastic resonance, respectively. An example of the latter subdivided group is the ultra-low-frequency interannual variation in Lorenz’s “GCM”.

There are a couple of important issues in modeling climate and climate variations; governing equations and climatic predictability.

Generally numerical models used in the field of GFD and CD contain some processes which are described with empirical formulae. For example, the piecewise linear function (2.3) for $a(T)$ is a very crude assumption and can not be deduced from basic physical laws. Even in a sophisticated GCMs some sub-grid scale parameterizations are formulated inductively, which might not be appropriate for a different climate from the present state. As pointed in [40] the detailed dynamical properties of these models (particularly simple models) may not have anything to do with the properties of the real-life system. However, simple models could be useful to get conceptual understanding of the complicated climate system and to illustrate the basic process of the system, although they could not be useful for future climate predictions.

Even if we had developed a perfect climate prediction model, there is a limit of predictability due to the chaotic nature of the earth’s climate system. However, it might be possible to predict a change in time-averaged and area-averaged quantities or to estimate the climate change due to the change in external forcings. It is important to know what is predictable and what else is not in principle. Anyhow for the purpose of climate prediction it is important to develop more reliable GCMs without parts obtained inductively from observational data.

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