A Numerical Study on Bifurcation Properties of Some Low-Order Models in Geophysical Fluid Dynamics

Shigeo Yoden

Geophysical Institute, Kyoto University

Low-order models are systems of ordinary differential equations which are simplified by extreme reduction of the number of dependent variables in spectral models. In the field of geophysical fluid dynamics many kinds of low-order models have been developed for these twenty years. The low-order models are one of the feasible procedures to investigate the nonlinear characters of the original partial differential equations. In this article a quasi-geostrophic, barotropic, low-order model with six variables is constructed to investigate the bifurcation properties of steady solutions and time-dependent solutions in a numerical model of geophysical fluid system.

The six-variables model has the multiplicity of the stable solutions (steady and periodic solutions). The multiplicity of steady solutions is possible due to the existence of critical points (i.e., limit points and bifurcation points). Periodic solutions appear by super-critical or sub-critical Hopf bifurcations. Time-dependent solutions, that is, periodic and chaotic (=non-periodic) solutions have some characteristics: Symmetric saddle-node bifurcation, period doubling bifurcation and intermittent chaos.

I. INTRODUCTION

All kinds of fluid motion are governed by nonlinear partial differential equations. On several occasions the nonlinearity has a crucial role in determining the characteristics of fluid motion. In the field of geophysical fluid dynamics, the concept of "multiple flow equilibria" has become of major interest as an example of nonlinear phenomena, because the atmosphere and oceans have some bimodal characters such as blocking phenomena and Kuroshio meander (see e.g., Wallace & Blackmon and Robinson & Taft). The multiplicity of equilibrium solutions can be analyzed with the bifurcation theory of steady solutions in nonlinear systems. Another interesting phenomenon is non-steady (periodic and chaotic) states observed in laboratory experiments and in the real atmosphere and oceans. In the rotating annulus experiments with horizontal contrast of heating, stepwise transitions from laminar flow to turbulence are observed like as those in the experiments on Rayleigh-Bénard convection (see Hide & Mason). The stepwise transitions to turbulence can also be understood as bifurcation sequences of non-steady solutions.

Low-order models are systems of ordinary differential equations which have been simplified by extreme reduction of the number of dependent variables (Lorenz). The low-order models are one of the feasible procedures to investigate these nonlinear properties of the original partial differential equations. Lorenz system on the problem of Rayleigh-Bénard convection is a familiar low-order model for applied mathematicians as well as meteorologists (Sparrow). After the pioneering work by Lorenz, many kinds of
low-order models have been developed in the field of geophysical fluid dynamics (Shirer & Wells\textsuperscript{7}) because of their economical feasibility and analytical simplicity.

One of them is the quasi-geostrophic, barotropic model by Charney & DeVore\textsuperscript{31}. They considered a resonance mechanism of Rossby waves involving interactions with bottom topography in order to make a possible explanation of the blocking mechanism due to the multiple flow equilibria.

A series of works by the author are concerned with a direct extension of Charney & DeVore's model (Yoden & Hirota\textsuperscript{9}, Yoden\textsuperscript{10}) and with a similar problem in baroclinic fluid (Yoden\textsuperscript{11,12}). In this article, a low-order model with six variables, which is an extension of Charney and DeVore's model, is constructed and its bifurcation properties are investigated in detail (Most of the results are published in another paper by the author\textsuperscript{10}). This model includes barotropic process that zonal flow can be accelerated by the convergence of eddy momentum flux. Compared with the three-variables model in Charney & DeVore\textsuperscript{9}, some new bifurcation properties are obtained such as (1) bifurcation of steady solutions with symmetry breaking, (2) appearance of periodic solutions due to Hopf bifurcation, (3) symmetric saddle-node bifurcation, period doubling bifurcation and intermittent chaos. As for the multiplicity of stable solutions, there exist stable periodic solutions in addition to stable steady solutions for some given external parameters.

II. A LOW-ORDER MODEL WITH SIX VARIABLES

1. Model Description

Motions of rotating, homogeneous and incompressible fluid with large aspect ratio (horizontal length scale is much greater than vertical one) can be described by the shallow-water equations. If we direct our attention to motions which have time scale longer than that of rotation, the quasi-geostrophic approximation is permissible and the non-dimensionalized potential vorticity equation is obtained as follows (see Pedlosky\textsuperscript{10}):

\[
\frac{\partial}{\partial t}(p^2 - F)\psi + J(\psi, p^2\psi + \beta y + h) = -k p^2\psi + kp^2\psi^*,
\]

where \(\psi(x,y,t)\) is a stream function for the O(1) velocity field, \(x\) 'eastward' direction, \(y\) 'northward' direction, \(t\) time, \(p^2\) horizontal Laplacian, \(J(a,b)\) horizontal Jacobian, \(F\) rotational Froude number, \(h\) height of the bottom topography, \(\beta\) latitudinal variation of Coriolis parameter, and \(1/k\) time constant for dissipation and forcing. In (1) the \(\beta\)-plane approximation was also taken and the bottom topography was assumed to be small amplitude compared with the fluid depth. The first term in the right hand side is the dissipation in the Ekman layer and the second one is a vorticity source which corresponds to the thermal forcing for the generation of background flow in the atmosphere.

If we expand \(\psi, \psi^*\) and \(h\) in orthogonal functions \(\{\phi_k(x,y)\}\) and substitute them into Eq. (1), we finally obtain a spectral form of the potential vorticity equation (1) as follows:

\[
\frac{d}{dt}\phi_i = (a_i^2 + F)^{-1}\left\{\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{ijk}\phi_j\psi_k(-a_k^2\psi_k + h_k) + \beta \sum_{j=1}^{\infty} b_{ij}\phi_j\psi_k - ka_k^2\phi_i + ka_k^2\phi_i^*\right\},
\]

where \(a_i^2, b_{ij}\) and \(c_{ijk}\) satisfy following relations.

\[
p^2\phi_i = -a_i^2\phi_i,
\]
Bifurcation Properties in Geophysical Fluid Dynamics

\[
\frac{\partial \phi_j}{\partial x} = \sum_{i=1}^{\infty} b_{ij} \phi_i, \quad b_{ij} = \frac{\partial \phi_j}{\partial x},
\]

(4)

\[
J(\phi_j, \phi_k) = \sum_{i=1}^{\infty} c_{ijk} \phi_i, \quad c_{ijk} = \frac{\partial J(\phi_j, \phi_k)}{\partial x}.
\]

(5)

Here we consider a channel flow as in Lorenz\textsuperscript{14} and retained only six components of \( \{\phi_i\} \):

\[
\begin{align*}
\phi_A &= \sqrt{2} \cos 2y, \\
\phi_C &= \sqrt{2} \cos 2y, \\
\phi_K &= 2 \sin y \cos nx, \\
\phi_M &= 2 \sin 2y \cos nx, \\
\phi_L &= 2 \sin y \sin nx, \\
\phi_N &= 2 \sin 2y \sin nx.
\end{align*}
\]

(6)

If the vorticity source term is given by a zonal component \( \phi_A^* \phi_A \) and the bottom topography is restricted to a single wave component \( h_K \phi_K \), an autonomous system with six variables is obtained as follows:

\[
d\phi_A/dt = -k_{01} \phi_A - k_{01} \phi_A + h_{01} \phi_L, 
\]

(7)

\[
d\phi_K/dt = -k_{n1} \phi_K + \beta n_1 \phi_L - \alpha n_1 \phi_A \phi_L - \delta n_1 \phi_C \phi_N, 
\]

(8)

\[
d\phi_L/dt = -k_{n1} \phi_L - k_{n1} \phi_A - \beta n_1 \phi_K + \alpha n_1 \phi_A \phi_K + \delta n_1 \phi_C \phi_M, 
\]

(9)

\[
d\phi_C/dt = -k_{02} \phi_C + h_{02} \phi_N + \varepsilon_n (\phi_K \phi_N - \phi_L \phi_M), 
\]

(10)

\[
d\phi_M/dt = -k_{02} \phi_M + \beta n_2 \phi_N - \alpha n_2 \phi_A \phi_N - \delta n_2 \phi_C \phi_L, 
\]

(11)

\[
d\phi_N/dt = -k_{02} \phi_N - k_{02} \phi_C - \beta n_2 \phi_M + \alpha n_2 \phi_A \phi_M + \delta n_2 \phi_C \phi_K, 
\]

(12)

where \( \alpha_{nm}, \beta_{nm}, \delta_{nm}, \varepsilon_n, h_{nm} \) and \( k_{nm} \) are positive constants and full descriptions are given in Charney & DeVore\textsuperscript{5a}. These constant coefficients are dependent on the channel geometry (channel width and rotation rate), topography (height and wavenumber) and the property of fluid (eddy viscosity coefficient). In this study the intensity of the vorticity source \( \phi_A^* \) is varied bit by bit as a bifurcation parameter. Of course the bifurcation properties depend on above coefficients, but we use the same values of coefficients as those in Charney & DeVore\textsuperscript{5b} except for the height of topography.

Equations (7)\textsuperscript{1}~(12) have a symmetry with respect to the second \( y \)-mode: \( (\phi_A, \phi_K, \phi_L, \phi_C, \phi_M, \phi_N) \rightarrow (\phi_A, \phi_K, \phi_L, -\phi_C, -\phi_M, -\phi_N) \). If the second \( y \)-mode components are equal to zero, Eqs. (7)\textsuperscript{1}~(9) surrounded by a broken line constitute a subsystem of (7)\textsuperscript{1}~(12). Charney & DeVore\textsuperscript{5b} obtained the steady solutions in this sub-system and discussed the multiplicity of steady solutions. However, nonlinear interaction between the zonal flow and wave components is not incorporated in this most simplified system. At least two \( y \)-modes are necessary to depict the wave-zonal flow interaction. The set of ordinary differential equations with six variables under consideration includes such a process in Eq. (10): \( \varepsilon_n (\phi_K \phi_N - \phi_L \phi_M) \) is an acceleration of zonal flow \( \phi_C \) due to the convergence of wave momentum flux.

2. Steady Solutions and Their Linear Stability

The steady solutions of Eqs. (7)\textsuperscript{1}~(12) are obtained by setting the time derivatives equal to zero. Eliminating wave components \( \phi_K, \phi_L, \phi_M \) and \( \phi_N \), the equations are reduced into two equations about \( \phi_A \) and \( \phi_C \) (Eq. (35) in Charney & DeVore\textsuperscript{5b}). All of the
steady solutions in the range of $0 < \psi_A < \psi_A^*$ are obtained by a numerical procedure similar to that in Yoden11).

Figure 1 shows all of the steady solutions for $0 < \psi_A^* < 0.5$ and $h_K = 0.1$. Steady solutions (1)~(3) with zero second $y$-modes are corresponding to those obtained in Charney & DeVore8). There also exist eight steady solutions with both $y$-modes (4)~(11). A pair of steady solutions (4) and (5), which are symmetric about the second $y$-modes each other, branch off from (1) at $\psi_A^* = 0.104\cdots$ by a symmetry breaking bifurcation. A pair of (10) and (11) bifurcate from (2) at $\psi_A^* = 0.177\cdots$. There exist seven limit points in Fig. 1 at which a pair of the steady solutions coalesce and disappear with increasing or decreasing the parameter $\psi_A^*$. Steady solutions (4), (6), (8) and (10) (or (5), (7), (9) and (11)) are connected with each other through these limit points. The steady solutions (1) and (2) are also connected at further increased $\psi_A^*$.

Stability properties of the steady solutions are investigated by solving an eigenvalue problem of the linearized equations of (7)~(12). If we assume that the perturbation is in the form $\psi \exp(\sigma t)$, real part of the eigenvalue $\sigma$ gives the growth rate of perturbation. Stable solutions without positive $\sigma$ are the solutions (1) in $\psi_A^* < 0.093\cdots$, all of the solutions (3), the solutions (4) and (5) in $0.172\cdots < \psi_A^* < 0.44\cdots$, and the solutions (8) and (9) in $0.175\cdots < \psi_A^* < 0.181\cdots$. Therefore multiplicity of stable steady solutions are possible in the present six-variables model. There exist five steady solutions in the range of $0.175\cdots < \psi_A^* < 0.181\cdots$. On the other hand there is no stable solution in $0.093\cdots < \psi_A^* < 0.135\cdots$.

Figure 2 shows the stability diagram $(\sigma)$ in the neighborhood of the bifurcation point of (4) and (5) from (1) and the limit point of (8) and (10) (or (9) and (11)). Eigen-
Fig. 2. Linear stability of steady solutions in the neighborhood of bifurcation point (left) and limit point (right). Real part of eigenvalue ($\sigma_r$) is presented. Solid line shows pure real eigenvalues and dotted line shows complex conjugate eigenvalues.

values of both branches are equal at the bifurcation point and at the limit point, and one of the real eigenvalue becomes zero at these points. At the bifurcation point the eigenvector corresponding to the zero eigenvalue consists of the second $y$-mode components, which has the same symmetry as the bifurcated solutions (4) and (5). All of these results are consistent with the bifurcation theory of steady solutions (Chow & Hale\textsuperscript{15}, Matsuda\textsuperscript{16}).

The steady solution (1) becomes unstable with complex conjugate eigenvalues when the external parameter $\psi_A^*$ exceeds a critical value ($=0.093\cdots$) denoted by an arrow in Fig. 2. It is expected that periodic solutions appear by a Hopf bifurcation.

3. Time-Dependent Solutions

Numerical integrations of the ordinary differential equations of (7)~(12) are performed to elucidate the appearance of periodic solutions by a Hopf bifurcation and to investigate further bifurcation sequences of time-dependent solutions. Unless otherwise mentioned, we choose the steady solution (1) with small perturbations as an initial condition. Some of the time-dependent solutions are shown in Fig. 3, which is a projection of their trajectories onto $\psi_0-\psi_A$ plane. Bifurcation diagram of steady and time-dependent solutions are presented in Fig. 4 schematically. Here figures in round bracket denote steady solutions in Fig. 1 and those in circle denote the time-dependent solutions shown in Fig. 3.

When the external forcing parameter exceeds the critical value $\psi_A^*=0.093\cdots$, there
appears a symmetric (with respect to the second \( y \)-mode) periodic solution (1). This periodic solution has a small amplitude and a period of 608.8 in non-dimensional time, which is close to \( 2\pi/\alpha_t \) at the bifurcation point (=667.3). Note that the first \( y \)-mode components have the half period due to the symmetry. Because the stable periodic solutions bifurcate on the side of unstable steady solution, this is a case of super-critical Hopf-bifurcation.

As \( \phi_A^* \) increases, amplitude of the symmetric periodic solution becomes large and the period decreases a little (2, 3). At \( \phi_A^* = 0.155 \) (4), the trajectory is complex compared with (1–3) due to the higher harmonics of fundamental frequency, but the symmetry about the second \( y \)-mode is still hold.

The symmetric saddle-node bifurcation (Sparrow\(^b\)) takes place at \( \phi_A^* = 0.1559\ldots \). That is, two stable non-symmetric solutions appear when the pre-existing stable sym-
Fig. 4. Bifurcation diagram of steady solutions (upper part) and time-dependent solutions (lower part) for $h_3=0.1$. Symbol ▲ denotes a super-critical Hopf bifurcation and ●, ■ sub-critical Hopf bifurcations.

Symmetric periodic solution becomes unstable. One of the non-symmetric orbits is shown in Fig. 3-⑥ and the other is obtained by changing the sign of the second $y$-mode components. The period doubling bifurcation happens at $\phi_A^*=0.1639⋯(⑥)$. There exist two stable periodic solutions as well due to the second $y$-mode symmetry. With increasing $\phi_A^*$, further period doublings and transition into chaotic solutions do not take place but the period halving transitions take place at $\phi_A^*=0.1679⋯$ and $0.1714⋯(⑦)$.

There appear chaotic solutions in $0.1747⋯<\phi_A^*<0.1753⋯(⑧)$. Structure and characteristics of these chaotic solutions are investigated later in detail. Two periodic solutions around the stable steady solutions (4) or (5) are obtained in $0.1753⋯<\phi_A^*<0.1815⋯(⑨)$. At $\phi_A^*=0.172⋯$ the unstable steady solutions (4) and (5) become stable with pure imaginary (conjugate) eigenvalues. Therefore it is considered that these are a case of sub-critical Hopf bifurcation as schematically presented in Fig. 4. There exist three stable steady solutions (3), (4) and (5), and two stable periodic solutions in $0.1753⋯<\phi_A^*<0.1815⋯$, and besides the steady solutions (8) and (9) are also stable around $\phi_A^*=0.18$.

Some examples of the time evolution of zonal component $\phi_A$ are shown in Fig. 5 for periodic and chaotic solutions. Spectral analysis of the time evolution sequences is performed by using the FFT method and the power spectral density of $\phi_A$ component is also shown in Fig. 5. Symmetric saddle-node bifurcation and period doubling bifurcation take place between $\phi_A^*=0.155$ and $0.165 (④$ and $⑥)$. Spectral peaks of $f/4, f/2$ and their higher harmonics ($f$ is a fundamental frequency) are observed in the middle of Fig. 5. Chaotic solution for $\phi_A^*=0.175$ is characterized by its intermittent behavior. The difference of the power spectral density between periodic and chaotic solutions is clear in Fig. 5. Noise level increases from $\sim 10^{-6}$ to $\sim 10^{-4}$ in the power spectral density.

So far we have mentioned in detail about the bifurcation properties for a given topo-
Fig. 5. Time evolution of $\phi_A$ component in periodic and chaotic solutions (left) and power spectral density of $\phi_A$ variation (right).

Graphical height $h_K=0.1$. The properties depend on the topographic height as well as other fixed parameters in partially qualitatively and in partially quantitatively. Bifurcation properties of time-dependent solutions are investigated for several values of $h_K$. If the topographic height is small ($h_K < 0.06$), time-dependent solutions is attracted into one of the stable steady solutions, and periodic and chaotic solutions are not obtained. As $h_K$ increases, the value of $\phi_A^*$ at which the Hopf bifurcation takes place increases, the range of $\phi_A^*$ for periodic and chaotic solutions also increases, and the fundamental period decreases. For $k_K > 0.12$, chaotic solutions have some different properties compared with those for $h_K=0.1$. There exist some windows of periodic solutions in the chaotic regime of $\phi_A^*$. And there exist a case that chaotic solutions appear through the period doubling bifurcations for $k_K \geq 0.18$.

In order to obtain an information about the dynamics of the chaotic solutions, we take a Lorenz plot, which is a plot of $(M_n, M_{n+1})$ in $x$-$y$ plane where $M_n$ is the $n$-th local maximum value of $\phi_A$. In the Lorenz system (Lorenz 63, Sparrow 63), the Lorenz plot can be approximated by a graph of iterated mapping on an interval known as ‘baker’s trans-
formation.' Figure 6 shows the Lorenz plots for the chaotic solutions for $h_K = 0.12$. There exist some periodic windows between these two chaotic solutions of $\psi_4 = 0.215$ and $0.245$. Both of the Lorenz plots are not the mapping from $M_n$ to $M_{n+1}$ and have more complex structure than those in the Lorenz system. By comparison between the plots in Fig. 6, it is found that there is a structural difference in the chaotic solutions of two sides of the periodic windows. The Lorenz plot for $\psi_4 = 0.215$ has much prominence of its layered-sheet structure.

4. Discussion

An autonomous system of Eqs. (7)~(12) has the multiplicity of the stable solutions (steady and periodic solutions) in a certain range of external parameters. The multiplicity of steady solutions is possible under the existence of bifurcation points as well as limit points. In the present six-variables model, there exist periodic solutions, which appear by super-critical or sub-critical Hopf bifurcations. One of the eigenvalues of the linearized equations is equal to zero at bifurcation points and limit points and is pure imaginary at the Hopf bifurcation point in consistent with the bifurcation theory (Chow and Hale\textsuperscript{16}), Matsuda\textsuperscript{16}).

The bifurcation properties obtained in this study have some similarities with those in the Lorenz system (Lorenz\textsuperscript{51}, Sparrow\textsuperscript{49}): Symmetry breaking bifurcation of steady solution, Hopf bifurcation, symmetric saddle-node bifurcation, and period-doubling bifurcation. Chaotic behavior shown in Fig. 3-\textsuperscript{8} reminds us those in the Lorenz system. There exist two preferred 'climate' and sudden transitions take place irregularly. This is an example of the almost intransitive system defined by Lorenz\textsuperscript{17}). However, the chaotic solutions in the present system have some complicated dynamical structures compared with those in the Lorenz system as seen from the Lorenz plots in Fig. 6.

III. CONCLUDING REMARKS

The possibility of the multiple flow equilibria first pointed out by Charney and DeVore\textsuperscript{81} and some new bifurcation properties obtained in this six-variables model are
interesting in themselves as an example of nonlinear phenomena in a model of geophysical fluid system. However, it is important to point out that these results are obtained within the framework of low-order models. In order to clarify whether the obtained results are observed in a real geophysical fluid system, adequateness of the low-order models must be investigated. Namely, investigations about the effect of truncation and the justification of assumptions are needed as future problem (see Yoden\textsuperscript{18}).

In addition to the investigations with multi-variable models, laboratory experiments with a rotating tank will bring some interesting result about the nonlinear phenomena in the geophysical fluid system. Recently Masuda\textsuperscript{19} and Sakai\textsuperscript{20} independently showed the existence of multiple flow regimes in their laboratory experiments. They constructed rather different rotating tanks for mechanical understanding of the Kuroshio meander but both of them obtained two types of stable flow regimes for given external parameters. It is still uncertain that these experiments are directly connected with the real Kuroshio meander, but their results are very interesting in connection with theoretical works on multiple flow equilibria in geophysical fluid systems.

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