Gravity wave radiation from unsteady rotational flow in an $f$-plane shallow water system

Norihiko Sugimoto$^{a,b,*}$, Keiichi Ishioka$^a$, Shigeo Yoden$^a$

$^a$Department of Geophysics, Graduate School of Science, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

$^b$Department of Computational Science and Engineering, Graduate School of Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi 464-8601, Japan

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Abstract

Spontaneous gravity wave radiation from an unsteady rotational flow is investigated numerically in an $f$-plane shallow water system. Unlike the classical Rossby adjustment problem, where free development of an initially unbalanced state is investigated, we consider development of a barotropically unstable zonal flow which is initially balanced but maintained by zonal mean forcing. Gravity waves are continuously radiated from a nearly balanced rotational flow region even when the Froude number is so small that balance dynamics is thought to be a good approximation for the full system. The source of gravity waves is discussed by analogy with the theory of aero-acoustic sound wave radiation (the Lighthill theory). It is shown that the source regions correspond to regions of strong rotational flow. The gradual change of rotational flow causes gravity wave radiation. We propose an approximation for these strong sources on the assumption that the dominant flow in the jet region is non-divergent rotational flow. In addition, we calculate the zonally symmetric component of gravity waves far from the source regions, solving the Lighthill equation. Using scaling analyses for perturbations, these gravity waves can be calculated with only one approximated source term that is related to the latitudinal gradient of the fluid depth and the latitudinal mass flux.

*Corresponding author. Department of Computational Science and Engineering, Graduate School of Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi 464-8601, Japan.

E-mail address: sugimoto@fcs.coe.nagoya-u.ac.jp (N. Sugimoto).
In spite of its simplicity, this approximation not only explains the physical cause of gravity wave radiation, but gives an amount of source close to that obtained by classical approximation derived from vortical motion. © 2007 The Japan Society of Fluid Mechanics and Elsevier B.V. All rights reserved.

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1. Introduction

The purpose of this study is to investigate gravity wave radiation from a nearly balanced rotational flow. The motion of stratified rotational fluid has two distinct modes. One is balanced rotational mode of vortex, which evolves comparatively slow and has large scale in the horizontal direction. The other is unbalanced divergent mode of gravity wave, which, compared to the former mode, evolves relatively quickly and tend to have a small scale in the horizontal direction. Coupling of these modes are theoretically interesting (Hough, 1898; Longuet-Higgins, 1968). In the present study, we investigate the continuous gravity wave radiation from an unsteady jet flow, which is considered to be a nearly balanced rotational flow, using a numerical simulation of an $f$-plane shallow water system with forcing.

The mechanism of gravity wave radiation from a nearly balanced rotational flow is not fully understood in the atmospheric science. Gravity waves are ubiquitous in the atmosphere and play a fundamental role in driving the general circulation of the middle atmosphere by transporting significant amounts of energy and momentum (Holton et al., 1995). Although topography, convection, jets, fronts, cyclones and so on have been identified as origins of gravity waves (Fritts and Nastrom, 1992; Sato, 2000), the detailed source mechanism concerning how gravity waves are radiated remains an open problem, especially concerning non-orographic radiation of waves. It has been reported in observational studies that gravity waves are radiated from the polar night jet (Yoshiki and Sato, 2000), the sub-tropical jet (Kitamura and Hirota, 1989; Sato, 1994; Plougonven et al., 2003), and cyclones (Pfister et al., 1993; May, 1996). At the same time, it has been also reported in studies using global circulation models that strong gravity waves are observed near jet regions (Sato et al., 1999). However, these studies have reported the characteristics of radiated gravity waves only. The source of gravity waves and how gravity waves are radiated from rotational flows remain to be explained.

In the present study, we consider an idealized situation, in which gravity waves are radiated from barotropically unstable jet flow, and use a simplified model, $f$-plane shallow water system, for a numerical simulation. In the past decade, several numerical studies have been done to investigate gravity wave radiation from a life cycle of baroclinic instability. For example, O’Sullivan and Dunkerton (1995) studied gravity wave radiation from a baroclinic wave with a global circulation model. For example, O’Sullivan and Dunkerton (1995) studied gravity wave radiation from a baroclinic wave with a global circulation model, and showed that gravity waves were radiated from the exit region of jet streaks. Zhang (2004), using a mesoscale numerical simulation, obtained similar results. However, since the main goals of these studies are to investigate gravity wave radiation from particular geophysical phenomena, such as a baroclinic wave, many complicated processes that are not directly related to gravity wave radiation are included in the numerical models. In this study, we use an $f$-plane shallow water system, which is the simplest system in which both balanced rotational modes and unbalanced gravity wave modes exist. We consider a simple zonal jet flow, which is a steady state of $f$-plane shallow water system but barotropically unstable, and investigate gravity wave radiation from the unsteady jet flow.
In addition, here we adopt a forced-dissipative system to investigate continuous gravity wave radiation from rotational flows. Although Ford (1994) investigated gravity wave radiation from a vortex train in a similar system without forcing, his study was concerned with the radiation process in an early stage only. Thus, gravity waves were radiated from unsteady vortices, and further radiation ceased as the flow becomes stable. In contrast, we add a forcing to maintain zonal jet flow. The forcing maintains the unsteady jet flow, while barotropic instability causes the continuous gravity waves radiation. Since we start the numerical simulation from a balanced steady state with an infinitesimal disturbance, all gravity waves are considered to be radiated from the unsteady jet flow. Note that the forcing is added to maintain the zonal jet flow, not to generate gravity waves.

The gravity wave radiation process treated here does not correspond to the Rossby adjustment process (Rossby, 1938; Gill, 1977), in which gravity waves are radiated from an initial unbalanced state, but corresponds to the spontaneous radiation process which was recently proposed by Ford et al. (2000). This process is not an adjustment toward a balanced state, but an adjustment away from a balanced state. Vanneste and Yavneh (2004) investigated spontaneous gravity wave radiation from a balanced motion within the limit of small Rossby number with three-dimensional Boussinesq equations. Their results suggested the inevitability of gravity wave radiation, and hence the non-existence of an invariant slow manifold on which there are no gravity waves and the initial balance is never violated. Our study gives another example of the spontaneous gravity wave radiation process, and still more the process is continuous.

This study is fundamentally different from Ford (1994), since our system has a zonal forcing and thus gravity waves are continuously radiated from an unsteady jet flow. Furthermore, we focus on the source of gravity waves, and try to obtain an approximated source both in the near and far fields of the jet region. To calculate gravity wave amplitude in the far field from the source, we introduce a new equation which estimates gravity wave amplitude with high accuracy.

This paper is organized as follows. Section 2 describes basic equations and the setup of the nonlinear numerical experiment. Section 3 introduces the Lighthill equation for the forced-dissipative $f$-plane shallow water system in which the source of gravity waves is defined. In addition, solving this equation, we will show a way to calculate gravity waves far from the source region. The results of numerical experiments including the evolution of rotational balanced flows and gravity wave radiation from them are given in Section 4. Section 5 presents two types of approximation for the gravity wave source. One is for sources near the jet region, and the other is for sources that contribute to gravity waves far from the source region. Gravity wave radiation is discussed in Section 6 and conclusions are stated in Section 7.

2. Model description

The basic equations used in this study are shallow water equations on an $f$-plane with zonal jet forcing and zonal absorber dissipation. These equations are the simplest equations including both vortical motions and gravity wave motions. On an $f$-plane, which is a plane rotating with angular velocity $f/2$, where $f$ is the Coriolis parameter, the equations are given as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -g \frac{\partial \eta}{\partial x} - \tau(u - \bar{u}) - \beta(u - \bar{u}) ,$$

forcing absorber

(1)
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -g \frac{\partial \eta}{\partial y} - \alpha (v - 0) - \beta (v - 0), \tag{2}
\]

\[
\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} + (H_0 + \eta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\beta (\eta - \bar{\eta}), \tag{3}
\]

where dependent variables \( u \) and \( v \) are the longitudinal (\( x \)) and the latitudinal (\( y \)) velocities, respectively, and \( \eta \) is the free surface deviation from the mean depth of the fluid \( H_0 \). \( \bar{u} \) and \( \bar{\eta} \) are the basic state of \( u \) and \( \eta \), respectively, for the zonal jet defined below. External parameter \( g \) is the gravitational acceleration. The terms with relaxation parameters \( \alpha \) and \( \beta \) mean the effect of forcing to maintain the unstable zonal mean jet and the effect of absorber to dissipate gravity wave near the boundary, respectively. The former is added near the jet region only, while the latter is added near the boundary only. Note also that the jet forcing term acts as dissipation for non-zonal disturbances at around the jet region.

The basic state of the zonal jet, which is barotropically unstable, is prescribed as (Hartmann, 1983)

\[
\bar{u}(y) = u_0 \text{sech} \left\{ \frac{2(y - y_0)}{B} \right\}, \tag{4}
\]

where \( u_0, y_0, \) and \( B \) are the parameters to determine the intensity, the position, and the width of the jet, respectively. The free surface profile in the geostrophic balance with the zonal jet is

\[
\bar{\eta}(y) = -\frac{fB u_0}{g} \text{arctan} \left\{ \exp \left( \frac{2(y - y_0)}{B} \right) \right\}. \tag{5}
\]

In this study, the width \( B \) and the intensity \( u_0 \) are used as the length scale and the speed scale, respectively. Then the non-dimensional parameters \( Ro \) (Rossby number) and \( Fr \) (Froude number) are determined as

\[
Ro \equiv \frac{u_0}{fB}, \quad Fr \equiv \frac{u_0}{\sqrt{gH_0}}. \tag{6}
\]

We fix \( Ro = 100 \) and \( Fr = 0.3 \) for the numerical experiments, since we are interested in a case where a significant departure from a quasi-geostrophic system appears, and actually Sugimoto et al. (2007) observed a significant departure due to gravity wave radiation in the nonlinear phase of the instability for those parameter values. The Coriolis parameter and the width of the jet are also fixed as \( f = 1, B = \pi/10 \). Then, we have \( u_0 = 10\pi, \) and \( 1/\sqrt{gH_0} = 10\pi/0.3 \).

The domain for the numerical experiment is assumed to be periodic. The periods are \( 2\pi \) in longitudinal (\( x \)) direction, and \( 128\pi \) in latitudinal (\( y \)) direction. Fig. 1 shows the latitudinal profiles of the basic state for \( Ro = 100 \) and \( Fr = 0.3 \): the zonal flow \( \bar{u}(y) \), the free surface deviation \( \bar{\eta}(y) \), and the latitudinal gradient of the potential vorticity \( \partial q(y)/\partial y \) (\( q = (\zeta + f)/h; \) here \( \zeta = \partial v/\partial x - \partial u/\partial y \) is the vorticity and \( h = \eta + H_0 \) is the total depth of the fluid) in \( y \) direction. The latitudinal domain is set to be \( 1280 \) times of \( B \). Two jets of opposite directions are put at the positions of \( y_0 = 32\pi \) and \( 96\pi \), respectively, in order to satisfy the periodic boundary condition in \( y \) direction. Note that \( \bar{u} \) is symmetric with respect to the center of the jet while \( \bar{\eta} \) in the geostrophic balance is antisymmetric. That is, the total depth
Fig. 1. Latitudinal profiles of the basic state with $Ro = 100$ and $Fr = 0.3$ for the whole region (upper panels), and those magnified around the jet (lower panels). (left) The zonal flow $u(y)$, (center) the free surface displacement $\eta(y)$, and (right) the latitudinal gradient of the potential vorticity $d\tilde{q}(y)/dy$ in $y$ direction.

of the fluid is shallowest at $y = 64\pi$ and deepest at $y = 0, 128\pi$. Since $d\tilde{q}(y)/dy$ changes its sign around the jet, this basic flow satisfies a well-known necessary condition for the barotropic instability (Ripa, 1983).
For the numerical calculation, we rewrite the shallow water equations (1)–(3) for different dependent variables \((\zeta, \delta, \Phi)\), which are the vorticity, the divergence \((\delta = \partial u / \partial x + \partial v / \partial y)\) and the geopotential height \((\Phi = gh)\), respectively, which yields

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} &= -\frac{\partial (u \zeta)}{\partial x} - \frac{\partial (v \zeta)}{\partial y} - f \delta - \alpha (\zeta - \zeta) - \beta (\zeta - \bar{\zeta}), \\
\frac{\partial \delta}{\partial t} &= \frac{\partial (v \zeta)}{\partial x} - \frac{\partial (u \zeta)}{\partial y} + f \zeta - \nabla^2 (E + \Phi) - \alpha (\delta - 0) - \beta (\delta - 0), \\
\frac{\partial \Phi}{\partial t} &= -\frac{\partial (u \Phi)}{\partial x} - \frac{\partial (v \Phi)}{\partial y} - \beta (\Phi - \bar{\Phi}),
\end{align*}
\]

where \(\nabla^2\) is the horizontal Laplacian. \(E = (u^2 + v^2)/2\) and \((u, v)\) are obtained from the stream function \(\psi\) and the velocity potential \(\phi\) defined as \(\zeta = \nabla^2 \psi\) and \(\delta = \nabla^2 \phi\) \((u = -\partial \psi / \partial y + \partial \phi / \partial x, v = \partial \psi / \partial x + \partial \phi / \partial y)\). \(\bar{\zeta}\) and \(\bar{\Phi}\) are the vorticity and the geopotential of the basic state calculated from (4) and (5). Eqs. (7)–(9) are computed using a spectral transform method (Ishioka, 2002). Namely, the dependent variables are expanded as

\[
W(x, y, t) = \sum_{k=-K}^{K} \sum_{l=-L}^{L} s_{kl}(t)e^{ikx}e^{ily/64},
\]

where \(W(x, y, t)\) represents \(\zeta, \delta, \Phi\). \(K\) and \(L\) are the truncation wave numbers for \(x\) and \(y\) direction, respectively. Since the time evolution of \(\zeta, \delta, \Phi\) in physical space is written as

\[
\frac{\partial W}{\partial t} = Z(x, y, t),
\]

where \(Z\) represents the right-hand side of (7)–(9), respectively, the time evolution of the coefficient \(s_{kl}\) is determined by the forward transform as

\[
\frac{ds_{kl}}{dt} = \frac{1}{256\pi^2} \int_0^{128\pi} \int_0^{2\pi} Z(x, y, t)e^{-ikx}e^{-ily/64} \, dx \, dy.
\]

We set the domain size in \(y\) direction to be larger by using \(K = 21, L = 5376\) \((64 \times 16384\) grids), since we focus on gravity wave radiation from the jet region toward \(y\) direction. The fourth order Runge-Kutta method is used for time-integrations with an increment of 0.00015. Zonal jet forcing is introduced around the jet region \((28.5\pi \leq y \leq 35.5\pi, 92.5\pi \leq y \leq 99.5\pi)\) and the forcing parameter is fixed as \(\alpha = 8\). Absorbing layers are introduced near the boundary and the middle area between the two jets \((y \leq 12.5\pi, 51.5\pi \leq y \leq 76.5\pi, 115.5\pi \leq y)\) in order to dissipate gravity waves crossing over these regions. We also introduce an artificial viscosity term \(\nu (\nabla^2)^5 W\) to smooth numerical behavior, and fix the viscosity coefficient as \(\nu = 10^{-34}\).
3. The Lighthill equation for an f-plane shallow water system

In the following, the source of gravity waves is discussed by analogy with the theory of aero-acoustic sound wave radiation (Lighthill, 1952). We rewrite Eqs. (1)–(3) in flux form as

\[
\begin{align*}
\frac{\partial (hu)}{\partial t} + \frac{\partial (huu)}{\partial x} + \frac{\partial (huv)}{\partial y} - fhv + \frac{1}{2}g \frac{\partial h^2}{\partial x} + zh(u - \bar{u}) &= 0, \\
\frac{\partial (hv)}{\partial t} + \frac{\partial (hvu)}{\partial x} + \frac{\partial (hvv)}{\partial y} + fhv + \frac{1}{2}g \frac{\partial h^2}{\partial y} + zhv &= 0, \\
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} &= 0.
\end{align*}
\]

Here, we neglect the absorber terms since we focus on the source of gravity waves near the jet region. From (13)–(15), we obtain the Lighthill equation in the forced-dissipative system (see Appendix A),

\[
\left( \frac{\partial^2}{\partial t^2} + f^2 - c_0^2 \nabla^2 \right) \frac{\partial h}{\partial t} = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} - F, \tag{16}
\]

where \( x_1 = x, x_2 = y \) and \( c_0 = \sqrt{gH_0} \) denotes the phase speed of the fastest gravity wave. The key point on the Lighthill theory is to assume that the source term \( T_{ij} \) is only non-zero over a small enough region that the right-hand side of (16) may be approximated by a quadrupole point source, and further assume that the source flow, and hence \( T_{ij} \) may be regarded as known in terms of vortex dynamics. Then, we can compute the source term without knowledge of the wave field (Ford, 1994). Both assumptions are valid in the limit of small \( Fr \), when the vortical flow is governed by the equations for a two-dimensional incompressible fluid, and the gravity wave length are \( O(Fr^{-1}) \) larger than the scale of the vortical flow (Ford et al., 2000). Here we regard the left-hand side representing gravity wave propagation, and the right-hand side representing the source of gravity waves. Using the Einstein summation convention, \( T_{ij} \) and \( F \) are written as

\[
T_{ij} = \frac{\partial (hu_i u_j)}{\partial t} + \frac{f}{2} (\varepsilon_{ik} h u_j u_k + \varepsilon_{jk} h u_i u_k) + \frac{g}{2} \frac{\partial}{\partial t} (h - H_0)^2 \delta_{ij}, \tag{17}
\]

\[
F = \frac{\partial}{\partial t} \left( \frac{\partial h(u - \bar{u})}{\partial x} + \frac{\partial (hv)}{\partial y} \right) + zf \left( \frac{\partial (hv)}{\partial x} - \frac{\partial h(u - \bar{u})}{\partial y} \right). \tag{18}
\]

Here, \( \varepsilon_{12} = \varepsilon_{21} = -1, \varepsilon_{11} = \varepsilon_{22} = 0 \) and \( u_1 = u, u_2 = v \).

Next, we calculate zonally \((x)\) averaged temporal variation of \( h \) \((\partial h/\partial t)\) far from the source region solving (16) (see Appendix B),

\[
\frac{\partial \overline{h}(y, t)}{\partial t} = \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y_-}^{y_+} dy' J_0 \left( f \sqrt{\left( t - t' \right)^2 - \left( \frac{y - y'}{c_0} \right)^2} \right) N(y', t'). \tag{19}
\]
where \( y_\pm = y \pm c_0(t - t') \). \( \bar{N} \) means the source term averaged in \( x \) direction, which is defined as \( \bar{N}(y', t') = \partial^2 \mathcal{T}_{zz}(y', t')/\partial y'^2 - \mathcal{T}(y', t') \). Integrating the right-hand side of (19) yields

\[
\frac{\partial \overline{h}(y, t)}{\partial t} = \frac{1}{2c_0} \int_{t_0}^{t} dt'[\mathcal{T}(y_+, t') - \mathcal{T}(y_-, t')] + D,
\]

(20)

where \( \mathcal{T} \) is a function which satisfies \( \bar{N}(y, t) = \partial \mathcal{T}(y, t)/\partial y \) and \( D \) means the effect of dispersion. If we know the source term near the jet, we can estimate \( \partial \overline{h}/\partial t \) far from the jet region from (20). Note that we will omit the contribution from the dispersion term \( D \) in Section 4.

4. Results

In this section, we will show the results of the numerical experiment. Neglecting the third order of disturbances, we define the energy for each zonal wave number as

\[
E_k \approx \frac{1}{2} \{ \varphi_0(u_k^2 + v_k^2) + \varphi_k^2 \},
\]

(21)

where the subscript \( k \) means zonal wave number. Fig. 2 shows the time evolution of the total energy for zonal wave number \( (k = 1–8) \). In the linear phase of the barotropic instability \( (t \leq 1) \), the energy for each wave number grows exponentially with time. On the other hand, in the nonlinear phase \( (t \geq 1) \), the energy for each wave number changes aperiodically, showing chaotic behavior.

Fig. 3 shows the time evolution of the nonlinear interactions of each zonal wave component and the decay by the forcing that contribute to the energy change of zonal wave number \( k = 2 \). In the nonlinear phase, the dominant contributions to the energy of zonal wave number \( k = 2 \) are the interaction between \( k = 0 \) and 2 component (red line) and the effect of forcing (blue line). The time evolution of the energy of zonal wave number \( k = 2 \) is explained as follows. Since the zonal jet is barotropically unstable, the energy

![Fig. 2. Time evolution of the total energy for each zonal wave number (k = 1–8).](image-url)
Fig. 3. Time evolution of the nonlinear interaction of each zonal wave component and the decay by the forcing that contribute to the energy change of zonal wave number \( k = 2 \) (left) and those magnification (right). Each line denoted as ‘two numbers’ is the nonlinear wave–wave interactions of each zonal wave number. For example, red line denoted as ‘02’ means the interactions between \( k = 0 \) component and \( k = 2 \) component. Blue line denoted as ‘f2’ means the energy decay by the effect of forcing and black line denoted as ‘sum’ is the summation of these contributions.

Fig. 4. The \((t, y)\) cross-section of \( \partial h/\partial t \) for \( 6 \leq t \leq 7 \) and \( 16\pi \leq y \leq 48\pi \). Negative regions are painted pink. The blue lines denoted as ‘gr’ indicates the phase speed of the fastest gravity wave. Solid contour lines correspond to positive values and zero, while broken ones to negative values.

of zonal wave number \( k = 2 \) grows exponentially at first \((t \leq 1.7)\). When unstable mode of \( k = 2 \) grows to have a finite amplitude, this zonal jet is stabilized and further growth is suppressed \((t \geq 1.7)\). Then the energy of \( k = 2 \) decays due to the effect of forcing \((t \leq 3.6)\). Consequently, the zonal jet is destabilized by the forcing and the energy of \( k = 2 \) grows exponentially again. This shift of the growth and the decay leads to the quasi-periodic energy change of \( k = 2 \) component.

The \((t, y)\) cross-section of \( \partial h/\partial t \) in the nonlinear phase is shown in Fig. 4 \((6 \leq t \leq 7, 16\pi \leq y \leq 48\pi)\). \( \partial h/\partial t \) is a good index of gravity wave, since it indicates linear gravity wave propagation if there is no
source term as shown in (16). As the speed of the jet varies quasi-periodically in time, gravity waves are radiated continuously from the jet region (around \( y = 32\pi \)) along the phase speed of the fastest gravity wave, \( \sqrt{gH_0} = 100\pi/3 \) (blue line). Wave lengths of these gravity waves are much longer than the width of the jet.

Fig. 5 shows an example of the time evolution of the vorticity field and the geopotential height field. Fig. 6 shows \( \partial h/\partial t \) and the total source which is the right-hand side of (16) at the same timing as Fig. 5. At \( t = 6.05 \), regions of strong vorticity exist around \((x, y) = (0.3\pi, 32.2\pi)\), and these regions correspond to the regions of strong gravity wave source. As the magnitude of vorticity in these regions decays \((t = 6.15, 6.25)\), the corresponding source amplitude of gravity waves also becomes weaker. Then another strong vortex motion appears around \((x, y) = (1.3\pi, 31.8\pi)\) at \( t = 6.25 \), and these regions correspond to the new source regions \((x, y) = (1.7\pi, 31.8\pi)\) at \( t = 6.35 \). In the geopotential height field, regions of the strong source correspond to those of shallow geopotential height, because the vorticity and the geopotential height are mainly in the cyclostrophic balance in this parameter range \((Ro = 100, Fr = 0.3)\). In \( \partial h/\partial t \) field, as the source changes gradually with time, gravity waves are radiated continuously from regions of strong vorticity. Time evolution of the vorticity field causes that of the source regions, and the frequency of the radiated gravity waves is basically determined by that of the variation in the vorticity field. We can see gravity waves of wave number 2 in \( x \) direction near the source regions corresponding to the wave number of the source, while far from the source regions, gravity waves have no wavy structure in \( x \) direction. The reason is as follows. In this system, the dispersion relationship of gravity
Fig. 6. Time evolution (at the same timings as Fig. 5) of $\partial h / \partial t$ (contour) and the total source (color) which is the right-hand side of (16). The horizontal axis is $x$ ($0 \leq x \leq 2\pi$) and the vertical axis is $y$ ($24\pi \leq y \leq 40\pi$). Blue color shows source of negative sign, while red color shows that of positive sign. Contour lines are $0$ and $\pm 10^i$ ($i : -3, -2, \ldots, 8$). Solid contour lines correspond to positive values and $0$, while broken contour lines correspond to negative values.
The first term on the right-hand side of (23) is very small, considering that the angular frequency of the vortex motion near the jet region in Figs. 2 and 3 is \( \omega = 2\pi v \sim 27 \) (here \( v \) is the frequency) and \( gH_0 = (100\pi/3)^2 \) from (6). Therefore, zonal wave number \( k = 2 \) in \( x \) direction makes \( l^2 \) negative in (23). This is why gravity waves of zonal wave number \( k = 2 \) cannot propagate far from the source regions.

Since gravity waves far from the jet region are approximated to have no wavy structure in \( x \) direction, we focus on zonal \( k = 0 \) component of gravity waves only. Fig. 7 shows time variation of \( \partial h/\partial t \) far from the jet region at 11 values of \( y \) and that obtained by theoretical calculation from (20) omitting the dispersion term. Again, gravity waves propagate from the jet region along the phase speed of the fastest gravity wave (red line). Though some reflection of gravity waves occurs near the boundary, the period and the amplitude are estimated accurately by the calculation of zonally (\( x \)) averaged Lighthill equation. This result clearly shows that gravity waves radiated in our numerical experiment are properly resolved.
Fig. 8 shows time evolution of the latitudinal gradient of zonally averaged gravity wave source (17). Since the integration in (20) is carried out along the phase speed of gravity wave (blue line), if the line goes through the negative source regions, the value of $\frac{\partial h}{\partial t}$ becomes negative along the line.

5. Approximation of gravity wave source

5.1. Source near the jet region

We try to deduce an approximation for the gravity wave source near the jet region on the assumption that the flow is nearly non-divergent. Furthermore, we neglect forcing $z$, Coriolis parameter $f$, and $\frac{\partial h}{\partial t}$ term, since we focus on the source due to rotational flow. We also assume that $|u| \gg |v|$. Then Eq. (16) can be approximated to (see Appendix C)

$$
\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} - F \approx 2hu \frac{\partial}{\partial x} \left( \frac{\partial v \partial u}{\partial x \partial y} \right).
$$

(24)

Fig. 9 shows an example of the total source and the approximated source (24) at $t = 6.05$. We can see the pattern of the total source is approximated well by (24). This result shows that rotational component of the flow mainly contributes to the gravity wave source.

5.2. Source approximations for gravity waves far from the jet region

In the previous subsection, we have introduced the approximated source near the jet region on the assumption that the dominant flow is non-divergent. However, calculation from this approximated source
cannot estimate the amplitude of gravity waves far from the source region accurately (not shown). We need another approximation for the gravity waves far from the jet region.

5.2.1. Source approximation related to the latitudinal gradient of the depth

In this subsection, we try to deduce another approximated source which is suitable for the calculation of the zonal \( k = 0 \) component of gravity waves far from the jet region. We neglect dispersion term in (20), since the depth of the fluid is so deep that \( f^2 \) term in (22) can be neglected. Next, we rewrite \( T_{22} \) by using (1) and (3) as

\[
T_{22} \approx \frac{\partial(h v^2)}{\partial t} \\
= \frac{\partial h}{\partial t} v^2 + 2h v \frac{\partial v}{\partial t} \\
= \left( \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} \right) v^2 + 2hv \left( -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - fu - \frac{\partial (gh)}{\partial y} - xv \right). \tag{25}
\]

Dividing dependent variables \((u, v, h)\) into zonal basic state \((\bar{u}(y), 0, \bar{h}(y))\) and deviation \((u', v', h')\), (25) can be approximated as

\[
T_{22} \approx 2\bar{h}v' \left( -\bar{u} \frac{\partial v'}{\partial x} - fu' - \frac{\partial (gh')}{\partial y} - xv' \right), \tag{26}
\]

where we have neglected third-order terms with respect to the deviation. The first term in the right-hand side disappears if it is averaged in \( x \) direction. Therefore we apply scaling analysis for remaining three terms. Setting \( U', L', \) and \( H' \) as the characteristics velocity, length, and fluid depth scales for the deviation,
we find

$$T_{22} \approx 2\hbar v' \left( -f U' u^\dagger - \frac{g H' \partial h^\dagger}{L'} \frac{\partial y^\dagger}{\partial y} - x U' v^\dagger \right),$$

(27)

where the variables denoted by $^\dagger$ are $O(1)$. Using the scale law, $g H' / f L' U' \sim Ro$, for balanced perturbations when $Ro$ is high (Sugimoto et al., 2007), we have

$$T_{22} \approx f U' 2\hbar v' \left( -u^\dagger - Ro \frac{\partial h^\dagger}{\partial y^\dagger} - \frac{x}{f} v^\dagger \right).$$

(28)

Considering $Ro = 100$, $x = 8$, and $f = 1$ in this experiment, we obtain

$$T_{22} \approx -2\hbar \frac{\partial h}{\partial y}.$$  

(29)

This approximation means that the term related to the latitudinal gradient of the fluid depth times the latitudinal mass flux can be the source of the zonal component of gravity waves. Fig. 10 is the same as Fig. 7 but calculated from (29). We have a good estimation of the zonal component of gravity waves from the approximated source (29).

5.2.2. Source approximation related to vortical motion

By analogy with the theory of aero-acoustic sound wave radiation, we introduce another approximation of the gravity wave source which is directly related to vortical motion. Crow (1970) showed if the Mach number ($M = U/c_a$; $U$ is flow speed and $c_a$ is the phase speed of sound wave) is low, the flow can be approximated as non-divergent. In our system, $Fr$ plays the role of $M$. Since we fix $Fr = 0.3$, which can
be considered low, let us approximate the flow as non-divergent, and again neglect forcing $\alpha$, Coriolis parameter $f$, and $\partial h/\partial t$ term in (16), which yields

$$
\left( \frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2 \right) \frac{\partial h}{\partial t} = h \frac{\partial}{\partial t} \left( \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} u_i u_j \right).
$$

(30)

If the flow is non-divergent, the following equation holds:

$$
\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} u_i u_j = \nabla \cdot (\omega \times u) + \nabla^2 \left( \frac{u^2}{2} \right),
$$

(31)

where $\omega = (0, 0, \zeta)$ is the vorticity vector, and $u = (u, v, 0)$ is the velocity vector. Howe (1975) calculated sound waves far from the source region, and showed that the contribution from the second term in (31) is $O(M^2)$ of the first term. His conclusion is that if $M$ is low, the main contribution to the sound waves far from the source region is the same as the vortex source introduced by Powell (1964). This implies that the source for gravity waves far from the source region can be approximated as

$$
\frac{\partial}{\partial t} h \nabla \cdot (\omega \times u).
$$

(32)

This approximation clearly shows that regions of strong vorticity $\omega$ can radiate gravity waves. In the present study, we call this approximation (32) as a classical approximation.

Fig. 11 is the same as Figs. 7 and 10 but calculated from the classical approximated source (32). We have a good estimation of the zonal component of gravity waves from this approximated source. Both of (29) and (32) can estimate the amplitude of gravity waves far from the source region approximately. Since there is no time derivative, our original approximated source (29) has a simpler form than the other
approximation (32). Thus the calculation from (29) is easier than that from (32). It is interesting that in spite of its simple form, the calculation from our approximated source (29) gives a comparable estimation with that from the classical approximated source (32).

6. Concluding remarks

Motivated by observational studies which suggest gravity wave radiation from rotational flows in the real atmosphere, we investigated continuous gravity wave radiation from an unsteady rotational flow numerically in an $f$-plane shallow water system with forcing and dissipation. Though Ford (1994) investigated gravity wave radiation from initial unstable vorticity strip in a similar system, his study treated only initial unsteady rotational flow using a non-forcing system. Instead of this numerical experimental setting, we used forcing to keep an unsteady rotational flow. Then, gravity waves were continuously radiated owing to the unsteady motion of nearly balanced rotational flow. The results suggest that gravity waves can be radiated from the jet regions in the real atmosphere, even in the parameter range in which the balance relation is thought to be maintained, though our model is too simple to apply to the phenomena in the real atmosphere.

The source of gravity waves was discussed by analogy with the theory of aero-acoustic sound wave radiation. We showed the regions of strong source corresponded to the regions of strong rotational flow. Gravity waves were continuously radiated by the time evolution of rotational flow in these source regions. The frequency of radiated gravity waves was basically determined by that of the unsteady jet. Though the source in the jet region had wave number 2 in $x$ direction, radiated gravity waves had no wavy structure in $x$ direction. This zonality of the radiated gravity waves was explained by the dispersion relationship. In the case that the frequency of the unsteady jet flow was low and the depth of the fluid was deep, gravity waves of wave number 2 could not propagate in $y$ direction. The zonal component of the radiated gravity waves was calculated accurately from the solution of the Lighthill equation by using the source in the jet region.

Furthermore, we proposed two approximations for the source of gravity waves. One approximation (24) was valid for the spatial pattern of the source near the jet region. This approximation was introduced on the assumption that the flow is non-divergent. This means that the spatial pattern of the source near the jet region is mainly caused by the rotational flow. However, this approximated source could not radiate zonal gravity waves directly to the far field. Apparent similarity of the spatial flow pattern over the source region was not the key to gravity wave radiation in this case.

The other approximation (29) was usable to estimate the amplitude of the zonal component of gravity waves far from the jet region. We applied the scale laws of balanced disturbances (Sugimoto et al., 2007) to the total source, and obtained the approximated source which is written in one term only. Since there is no time derivative term, our source approximation (29) has a simpler form and it is easier to calculate the amplitude of gravity waves than the classical approximation (32) which is introduced by analogy with the theory of the aero-acoustic sound wave radiation and directly related to vortical motion. In spite of its simplicity, this approximated source gave an estimation comparable to that with the classical approximation. In addition, this approximation allowed a reasonable physical explanation that the product of the latitudinal gradient of the fluid depth and the latitudinal mass flux causes gravity wave radiation.

There are several ways to extend our study to the phenomena in the real atmosphere. Since the shallow water system is introduced on the assumption of strong stratification, this system has only external gravity
waves which propagate in horizontal directions. In the real atmosphere, there are internal gravity waves which propagate both in horizontal and vertical direction. To investigate these internal gravity waves radiation process in the two-layer system will be interesting. Although further investigation is needed to approach general gravity wave radiation processes, similar theoretical expression of the gravity wave source, which was introduced in the present study, may be obtained in more generalized stratified models. It will be interesting to explore the possibility of a similar source approximation in spherical models, two-layer models, and the real atmosphere.

In this paper, we showed the results of one experiment only. Whether the approximations of gravity wave source can be applied to other parameter ranges or not is an interesting question. If Fr is low, since the flow is approximately non-divergent, sources near the jet region will be approximated in the same way and gravity waves far from the source region will be estimated by the same scaling analysis. If Fr is high, however, the flow cannot be approximated as non-divergent. Furthermore, if Ro is low, the scaling laws for the balanced perturbation for high Ro regimes are no longer valid (Sugimoto et al., 2007). Therefore, some contribution of other source terms must be considered. A series of parameter-sweep experiments are to be conducted in our next investigation. In addition, we can choose the forcing parameter to make the frequency of vortical motion high in the forced-dissipative system. In such experiments, gravity waves of another zonal wave number can be radiated. One of the our next steps will be to change the forcing parameter to investigate gravity wave radiation having non-zero zonal wave numbers.

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Appendix A.

The purpose of this appendix is to obtain the Lighthill equation in the forced-dissipative f-plane shallow water system. First, we take the divergence of (13) and (14), which yields

$$\frac{\partial}{\partial x} \left( \frac{\partial (hu)}{\partial t} \right) + \frac{\partial}{\partial y} \left( \frac{\partial (hv)}{\partial t} \right) + \frac{\partial^2 (hu^2)}{\partial x^2} + \frac{\partial^2 (hv^2)}{\partial y^2} + 2 \frac{\partial^2 (huv)}{\partial x \partial y} - f \left( \frac{\partial (hv)}{\partial x} - \frac{\partial (hu)}{\partial y} \right) + \frac{1}{2} g \left( \frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right) + \alpha \left( \frac{\partial (h(u - \bar{u}))}{\partial x} + \frac{\partial (hv)}{\partial y} \right) = 0. \quad (33)$$
Taking $\partial / \partial t$ of (33), we obtain
\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} \right) + \frac{\partial^2}{\partial x^2} \frac{\partial (hu^2)}{\partial t} + \frac{\partial^2}{\partial y^2} \frac{\partial (hv^2)}{\partial t} + 2 \frac{\partial^2}{\partial x \partial y} \frac{\partial (huv)}{\partial t} - f \frac{\partial}{\partial t} \left( \frac{\partial (hv)}{\partial x} - \frac{\partial (hu)}{\partial y} \right) + \frac{1}{2} \frac{g}{\partial t} \left( \frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right)
\]
\[\quad + \alpha \frac{\partial}{\partial t} \left( \frac{\partial (hu - \bar{u})}{\partial x} + \frac{\partial (hv)}{\partial y} \right) = 0, \tag{34}\]
where the term $(X)$ is rewritten by using the mean depth of the fluid $H_0$,
\[
\frac{1}{2} \frac{g}{\partial t} \left( \frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right) = gH_0 \nabla^2 \frac{\partial h}{\partial t} + \frac{1}{2} \frac{g}{\partial t} (h - H_0)^2 + \frac{1}{2} \frac{g}{\partial y^2} \frac{\partial^2}{\partial t} (h - H_0)^2. \tag{35}\]
The rotation of (13) and (14) and multiplication of $f$ lead to
\[
- f \frac{\partial}{\partial y} \left( \frac{\partial (hu)}{\partial t} + f \frac{\partial (hv)}{\partial t} \right) + f \frac{\partial^2 (hu^2)}{\partial x \partial y} + f \frac{\partial^2 (huv)}{\partial x \partial y} - f \frac{\partial^2 (huv)}{\partial x^2} + f \frac{\partial^2 (hu)}{\partial x \partial y}
\]
\[\quad + \alpha f \left( \frac{\partial (hv)}{\partial x} - \frac{\partial (hu - \bar{u})}{\partial y} \right) = 0. \tag{36}\]
Adding (34) to (36), we obtain the Lighthill equation for the forced-dissipative system as
\[
\left( \frac{\partial^2}{\partial t^2} + f^2 - c_0^2 \nabla^2 \right) \frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial (hu^2)}{\partial t} + f hv^2 + \frac{1}{2} \frac{g}{\partial t} (h - H_0)^2 \right)
\]
\[\quad + \frac{\partial^2}{\partial x \partial y} \left( 2 \frac{\partial (hu)}{\partial t} + f hv^2 - f hu^2 \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial (hv^2)}{\partial t} - f huv + \frac{1}{2} \frac{g}{\partial t} (h - H_0)^2 \right)
\]
\[\quad + \alpha \frac{\partial}{\partial t} \left( \frac{\partial (hu - \bar{u})}{\partial x} + \frac{\partial (hv)}{\partial y} \right)
\]
\[\quad + \alpha f \left( \frac{\partial (hv)}{\partial x} - \frac{\partial (hu - \bar{u})}{\partial y} \right). \tag{37}\]

Appendix B.

In this appendix, we will solve the Lighthill equation (16), written as
\[
\left( \frac{\partial^2}{\partial t^2} + f^2 - c_0^2 \frac{\partial^2}{\partial y^2} \right) A(y, t) = N(y, t), \tag{38}\]
under the initial and boundary conditions given by

\[
A(y, t_0) = \frac{\partial A(y, t_0)}{\partial t} = 0, \quad (39)
\]

\[
A(y, t) \to 0 \quad (y \to \pm \infty). \quad (40)
\]

First, let us express \(A\) and \(N\) by inverse Fourier transform as

\[
A(y, t) = \int_{-\infty}^{+\infty} \hat{A}(l, t) e^{ily} dl, \quad (41)
\]

\[
N(y, t) = \int_{-\infty}^{+\infty} \hat{N}(l, t) e^{ily} dl. \quad (42)
\]

Substituting these into (38), we obtain

\[
\left(\frac{\partial^2}{\partial t^2} + f^2 - c_0^2 l^2\right) \hat{A}(l, t) = \hat{N}(l, t). \quad (43)
\]

Defining \(\omega\) as

\[
\omega = l \sqrt{c_0^2 + f^2/l^2}, \quad (44)
\]

we can rewrite (43) as

\[
\left(\frac{\partial}{\partial t} + i\omega\right) \left(\frac{\partial}{\partial t} - i\omega\right) \hat{A}(l, t) = \hat{N}(l, t). \quad (45)
\]

Eq. (45) can be rewritten as

\[
e^{i\omega t} \frac{\partial}{\partial t} \left[ e^{-i\omega t} \left\{ e^{-i\omega t} \frac{\partial}{\partial t} (e^{i\omega t} \hat{A}(l, t)) \right\} \right] = \hat{N}(l, t). \quad (46)
\]

Integrating by \(t\) and using the initial condition (40), we obtain

\[
e^{-2i\omega t} \frac{\partial}{\partial t} (e^{i\omega t} \hat{A}(l, t)) = \int_{t_0}^{t} e^{-i\omega t'} \hat{N}(l, t') \, dt'. \quad (47)
\]

Again we integrating (47) by \(t\) and using the initial condition, which yields

\[
\hat{A}(l, t) = e^{-i\omega t} \int_{t_0}^{t} e^{2i\omega t'} \, dt' \int_{t_0}^{t'} e^{-i\omega t} \hat{N}(l, t') \, dt'. \quad (48)
\]
Therefore, we obtain

\[ A(y, t) = \int_{-\infty}^{+\infty} e^{iy} \int_{t_0}^{t} dt' \int_{t_0}^{t''} dt'' e^{i\omega(2t''-t-t')} \hat{N}(l, t') \]

\[ = \int_{-\infty}^{+\infty} e^{iy} \int_{t_0}^{t} dt' \int_{-(t-t')}^{t-t'} \frac{1}{2} d\xi e^{i\omega \xi} \hat{N}(l, t') \]

\[ = \int_{-\infty}^{+\infty} e^{iy} \int_{t_0}^{t} dt' \int_{-(t-t')}^{t-t'} \frac{1}{2} d\xi e^{i\omega \xi} \hat{N}(l, t') \]

\[ = \int_{-\infty}^{+\infty} e^{iy} \int_{t_0}^{t} dt' \frac{1}{2i\omega} (e^{i\omega(t-t')} - e^{-i\omega(t-t')}) \hat{N}(l, t') \]

\[ = \int_{-\infty}^{+\infty} e^{iy} \int_{t_0}^{t} dt' \frac{\sin(\omega(t-t'))}{\omega} \hat{N}(l, t'). \quad (49) \]

Since (49) is a convolution form, we can rewrite (49) as

\[ A(y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \int_{t_0}^{t} dt' R(\xi, t-t') N(y - \xi, t'), \quad (50) \]

where \( R \) is defined as

\[ R(y, t) = \int_{-\infty}^{+\infty} e^{iy} \frac{\sin(\omega t)}{\omega} \, dl \]

\[ = \int_{-\infty}^{+\infty} e^{iy} \frac{\sin \left( t \sqrt{f^2 + c_0^2 l^2} \right)}{\sqrt{f^2 + c_0^2 l^2}} \, dl \]

\[ = \int_{-\infty}^{+\infty} e^{i(y/c_0)l'} \frac{\sin \left( t \sqrt{f^2 + c_0^2 l'^2} \right)}{\sqrt{f^2 + c_0^2 l'^2}} \frac{1}{c_0} \, dl' \quad (51) \]

\[ = \begin{cases} 
\frac{\pi}{c_0} J_0 \left( f \sqrt{t^2 - (y/c_0)^2} \right) & \text{if } (y/c_0)^2 < t^2, \quad 0 < t, \\
0 & \text{if } (y/c_0)^2 > t^2, \\
-\frac{\pi}{c_0} J_0 \left( f \sqrt{t^2 - (y/c_0)^2} \right) & \text{if } (y/c_0)^2 < t^2, \quad t < 0. 
\end{cases} \quad (52) \]

Furthermore, changing independent variables, we obtain

\[ A(y, t) = \frac{1}{2c_0} \int_{-\infty}^{+\infty} d\xi \int_{t_0}^{t-|\xi/c_0|} dt' \frac{1}{2i\omega} (e^{i\omega(t-t') - (\xi/c_0)^2}) N(y - \xi, t') \]

\[ = \frac{1}{2c_0} \int_{-\infty}^{+\infty} dy' \int_{t_0}^{t-|y'/c_0|} dt' \frac{1}{2i\omega} (e^{i\omega(t-t') - (y'/c_0)^2}) N(y - y', t') \]

\[ = \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{c_0(t-t')}^{c_0(t-t')} dy' \frac{1}{2i\omega} (e^{i\omega(t-t') - (y'/c_0)^2}) N(y - y', t'). \quad (53) \]
By defining $y - y' = y''$, Eq. (53) can be rewritten as

$$A(y, t) = \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y+c_0(t-t')}^{y+c_0(t-t')} dy'' J_0 \left( f \sqrt{(t - t')^2 - ((y - y'')/c_0)^2} \right) N(y'', t')$$

$$= \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y-c_0(t-t')}^{y+c_0(t-t')} dy' J_0 \left( f \sqrt{(t - t')^2 - ((y - y')/c_0)^2} \right) N(y', t').$$

(54)

This is the same equation as (19). By defining $I$ to satisfy

$$N(y, t) = \frac{1}{p_106} I(y, t),$$

(55)

we can integrate the right-hand side of (54) by $y$, which yields

$$A(y, t) = \frac{\partial}{\partial y} \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y-c_0(t-t')}^{y+c_0(t-t')} dy' J_0(f S_q) I(y', t')$$

$$= \frac{1}{2c_0} \int_{t_0}^{t} dt' [I(y + c_0(t - t'), t') - I(y - c_0(t - t'), t')]$$

$$+ \int_{y-c_0(t-t')}^{y+c_0(t-t')} dy' \frac{(y - y')}{c_0^2 S_q} J_0'(f S_q) I(y', t')]$$

(56)

where $S_q = \sqrt{(t - t')^2 - ((y - y')/c_0)^2}$. Rewriting the differential of Bessel function $J_0$, we obtain

$$A(y, t) = \frac{1}{2c_0} \int_{t_0}^{t} dt' [I(y + c_0(t - t'), t') - I(y - c_0(t - t'), t')]$$

$$+ \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y-c_0(t-t')}^{y+c_0(t-t')} dy' \left( \frac{f^2}{c_0^2} (y - y') \right) \frac{1}{2} J_0(f S_q) I(y', t')$$

$$+ \frac{1}{2c_0} \int_{t_0}^{t} dt' \int_{y-c_0(t-t')}^{y+c_0(t-t')} dy' \left( \frac{f^2}{c_0^2} (y - y') \right) \frac{1}{2} J_2(f S_q) I(y', t').$$

(57)

This is the same equation as (20) except that the dispersion term $D$ in (20) is expressed explicitly here.

Appendix C.

Let us approximate the right-hand side of (16) on the assumption that the flow is non-divergent. Since we focus on the source due to rotational flow, we neglect the terms related to $\alpha, f$, and $\partial h/\partial t$. 
The approximation yields

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} - F \approx \frac{\partial^2}{\partial x^2} \left( \frac{\partial (hu^2)}{\partial t} \right) + \frac{\partial^2}{\partial x \partial y} \left( 2 \frac{\partial (hu v)}{\partial t} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial (hv^2)}{\partial t} \right)
\]

\[
= h \frac{\partial}{\partial t} S + \frac{\partial^2}{\partial x^2} \left( u^2 \frac{\partial h}{\partial t} \right) + \frac{\partial^2}{\partial x \partial y} \left( 2uv \frac{\partial h}{\partial t} \right) + \frac{\partial^2}{\partial y^2} \left( v^2 \frac{\partial h}{\partial t} \right)
\]

\[
\approx h \frac{\partial}{\partial t} S.
\] (58)

Here, assuming that the flow is non-divergent, we can simplify \( S \) as

\[
S = \frac{\partial^2}{\partial x^2} (u^2) + \frac{\partial^2}{\partial x \partial y} (2uv) + \frac{\partial^2}{\partial y^2} (v^2)
\]

\[
= \frac{\partial}{\partial x} \left( 2u \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial x \partial y} (2uv) + \frac{\partial}{\partial y} \left( 2v \frac{\partial v}{\partial y} \right)
\]

\[
= \frac{\partial}{\partial x} \left( -2u \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial x} \left( 2u \frac{\partial v}{\partial y} + 2v \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( -2v \frac{\partial u}{\partial x} \right)
\]

\[
= \frac{\partial}{\partial x} \left( 2u \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( -2v \frac{\partial u}{\partial x} \right)
\]

\[
= 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y}.
\] (59)

Therefore,

\[
\frac{1}{2} \frac{\partial S}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right)
\]

\[
= \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial t} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial t} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right).
\] (60)

Rewriting the term of time derivative of \( u \) and \( v \) by using the shallow water equations, and neglecting the terms related to \( z, f \), and \( \hat{h} / \hat{t} \) again, we obtain

\[
\frac{1}{2} \frac{\partial S}{\partial t} = \frac{\partial}{\partial x} \left( -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} \left( -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x}
\]

\[
- \frac{\partial}{\partial y} \left( -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} \right) \frac{\partial u}{\partial y} - \frac{\partial}{\partial x} \left( -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x}
\]

\[
= u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial t} \frac{\partial u}{\partial y} \right) - v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial t} \frac{\partial u}{\partial x} \right).
\] (61)

Assuming that \(|u| \gg |v|\), we neglect the second term in the most right-hand side of (61) compared with the first term. Then, we finally obtain

\[
\frac{1}{2} \frac{\partial S}{\partial t} \approx u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial t} \frac{\partial u}{\partial y} \right).
\] (62)
References